IGR midterm

Upload the answers on moodle before you leave. You may add to your answers later, and I may add to the questions by tomorrow. Any oral exam will use this only as a starting point for a conversation, in which the main thing I will look for beyond knowledge of content is fluency in applying it, in particular in a range of examples.

Unless specified otherwise all representations are of finite groups and finite dimensional over $\mathbb{C}$. A standing instruction for all problems is to explain when this situation can be can be relaxed and give most general version you can.

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0. Make two lists: (i) What you consider are the pivotal results of basic character theory. (ii) Quick and often used consequences of these. You should be able to prove all listed results on the spot in an oral exam. (Assume algebra of direct sums and Schur's lemma as part of the background.)

1. Write the full character tables of S_3 and S_4. No need to justify anything, but the meaning of everything should be clear. Then decompose the following representations of S_4. Here Std_n is the standard representation of S_n and W is the irreducible 2-dimensional representation of S_4.

   (i) $\bigwedge^2(Std_4) \otimes W$ (ii) $Ind(Sym^2(Std_3))$

2. For $H < G$, a kH-module W and a kG-module V prove that $Ind_H^G(W \otimes Res_H^G V)$ and $(Ind_H^G W) \otimes V$ are isomorphic kG-modules.

3. For the finite field $\mathbb{F}_q$ of cardinality q, Let $G =$ the group of automorphisms of affine line $\mathbb{F}_q^1$, i.e., of maps $x \mapsto ax + b$ where $a, b \in \mathbb{F}_q$ with $a \neq 0$. Show that G is a semidirect product of the subgroup $x \mapsto ax$ acting on the normal subgroup of $x \mapsto x + b$ of translations. It is a fact (check later) that there are G has q conjugacy classes. Construct as many irreps of G as you can. What are the dimensions of all of them?

4. *An extreme situation in finite characteristic p.* Show using Clifford's theorem that the only irreducible representation V of a group G of order p^n over a field k of characteristic p is trivial. (If k is finite, you can just use the set action of G on $V \setminus \{0\}$ to get a fixed point. If k is infinite, you can still reduce to the finite situation. The point here is to see an application of Clifford's theorem.)

5. *Multilinear algebra.* (i) Recall the algebra isomorphism $Sym(V \oplus W) \simeq Sym(V) \otimes Sym(W)$. Find the image of $Sym^d(V \oplus W)$ in the RHS. This is called the exponential property of the symmetric powers. (There is an analogous one for exterior powers too.)

   (ii) Recall the problem to show $\bigwedge(V^*) \simeq (\bigwedge(V))^*$ for a finite dimensional k-vector space V. This is dumb as stated (as some you saw) because both sides are vector spaces of dimension 2^n. Show that they are isomorphic as representations of $GL(V)$. Moreover one can transport the structure of algebra on LHS along the isomorphism to get an algebra strcture on the RHS. Can you make it explicit?

   Optional questions: the RHS is naturally a coalgebra, being the dual of an algebra. So the LHS becomes one too. Show that both sides are bialgebras (in fact Hopf algebras) and isomorphic as such.

6. *General nonsense.* Formulate a correct a tensor hom adjunction of the following form

   $$Hom_?(A \otimes B, C) = Hom_?(A, Hom_?(B, C))$$

   where B is an R-S bimodule and A, C carry the minimum structure (using rings R and S appropriately) that they need for this to be valid. Note that this is different from what we did in class, where A moved to the other side instead of B. You could make appropriate modifications in that, or work out a formal way to handle left v/s right, or work from basic principles, ...

   State the isomorphism (of what?) explicitly by making appropriate notational choices and *sketch* the essential points that need to be checked. Please spare me the gory details (but check them carefully in private at least once in life).

   If A, B, C are all bimodules, is there a version that incorporates this structure?