MIDTERM
THREE HOURS
100 MARKS
IF YOU USE ANY THEOREM/RESULT NOT PROVED IN THE CLASS, DO SUPPLY A COMPLETE PROOF

A. Prove or disprove the following statements with explanations. 60 Marks

1. Let χ be a non-trivial character mod q such that $L(1,\chi) = 0$. Then $L(s,\chi)$ has no other real zero $\sigma > 0$.

2. Let G be a finite group and H be a subgroup. The restriction map from the dual of G to that of H is onto.

3. Let $D(s) = 1 + \sum_{n>2} \frac{a_n}{n^s}$ be a Dirichlet series with 0 as the abscissa of convergence. Then $D(s)$ can have finitely many real zeros $\sigma > 1/2$.

4. $\sum p^{-\sigma}$ is asymptotic to $\log(1/(\sigma-1))$ as $\sigma \to 1^+$.

5. Let $f(s)$ be a non-constant entire function such that $f(s)\zeta(s)$ is a Dirichlet series for $\sigma > 1$ and with non-negative real coefficients. Suppose that $f(2/3) \geq 0$. Then $f(1) > 0$.

6. Let $q > 3$ and χ be a Dirichlet character mod q. Then $|\tau(\chi)| = \sqrt{q}$.

B. Let p be an odd prime number. Then show the following:
a) There exists a unique real non-trivial character χ mod p^4.
b) Is it true that there exists an integer $n > 1$, such that $p|n$ and the associated Gauss sum

$$\sum_{b \bmod p^4} \chi(b)\zeta_{p^2}^{bn}$$

is non-zero? Justify.
(40 marks)