Algebraic Geometry II Mid-semester Exam

02 March 2024

(1) Compute the cohomologies of the following spaces.

(a) $\mathbb{A}^3 \setminus \{\text{line}\}$,

(b) $\mathbb{A}^3 \setminus \{\text{point}\}$,

(c) More generally $\mathbb{A}^n \setminus \mathbb{A}^m$.

[3+4+5]

(2) Let $X = \mathbb{P}^n_k$.

(a) Compute all the cohomologies of the tangent sheaf $\mathscr{T}_X$.

(b) Let D be a hypersurface of degree d in X. Compute the cohomologies of $\mathcal{O}_D$.

(c) Let D be a hypersurface of degree $d \geq 2$ in X. Assume that D is nonsingular and $n > 2$. Compute the cohomologies $H^i(D, \Omega^1_D)$ for $i < n - 2$.

[2+2+8]

(3) Let X be an elliptic curve and K be the canonical divisor.

(a) Show that $\deg(K) = 0$ and K is linearly equivalent to the zero divisor.

(b) Let $\text{Pic}^o(X)$ denote the subgroup of $\text{Pic}(X)$ corresponding to divisors of degree 0. Let $P_0 \in X$ be a point. Then the map

$$X \longrightarrow \text{Pic}^o(X), P \longmapsto \mathcal{O}_X(P - P_0)$$

is a bijection.

[4+6]

(4) Let X be a curve of genus g.

(a) For an effective divisor D on X, show that

$$\dim|D| \leq \deg D.$$

Furthermore, equality holds if and only if $D = 0$ or $g = 0$.

(b) Show that there is a finite morphism $f : X \to \mathbb{P}^1$ of degree $\leq g + 1$.

[5+5]

(5) *Lüroth's Theorem*: Prove that if L is a subfield of a pure transcendental extension $k(t)$ of k, containing k, then L is also pure transcendental. (Hint: $K(\mathbb{P}^1_k) = k(t)$)

[4]

(6) *Automorphisms of a Curve of Genus* ≥ 2: Let X be a curve of genus $g \geq 2$ over a field of characteristic 0. Assume that the group $G = \text{Aut}(X)$ is finite. Let G have order n. Then G acts on the function field $K(X)$. Let L be the fixed field. Then the field extension $L \subseteq K(X)$ corresponds to a finite morphism of curves $f : X \to Y$.

Assume that f has degree n. Prove that n, the order of G is at most $84(g-1)$ using the following steps:

(a) If $P \in X$ is a ramification point, and $e_P = r$, show that $f^{-1}f(P)$ consists of exactly n/r points, each having ramification index r. Let $P_1, \ldots, P_s$ be a maximal set of ramification points of X lying over distinct points of Y, and let $e_{P_i} = r_i$. Then show that Hurwitz's theorem implies that

$$\frac{2g-2}{n} = 2g(Y) - 2 + \sum_{i=1}^{s} \left(1 - \frac{1}{r_i}\right).$$

(b) Since $g \geq 2$, the left hand side of the equation is > 0. Show that if $g(Y) \geq 0$, $s \geq 0$, $r_i \geq 2$, $i = 1, \ldots, s$ are integers such that

$$2g(Y) - 2 + \sum_{i=1}^{s} \left(1 - \frac{1}{r_i}\right) > 0,$$

then the minimum value of this expression is $1/42$. Then conclude that

$$n \leq 84(g-1).$$

[4+8]

(Hint: consider the cases $g(Y) = 0, 1$ and ≥ 2 separately.)