ALGEBRAIC GEOMETRY 1 FINAL EXAM

The max score is 60. The final question is a bonus question.

(1) (20 pts) Let Y be an integral noetherian scheme and n be a positive integer. Define the following
  - a (geometric) vector bundle V over Y of rank n.
  - A locally free $\mathcal{O}_Y$-module $\mathcal{F}$ of rank n.

  Show that there is a natural bijection $\mathcal{F} \leftrightarrow V$ between locally free $\mathcal{O}_Y$ modules of rank n and geometric vector bundles V over Y of rank n. Prove that this correspondence is functorial, i.e., a map between locally free $\mathcal{O}_Y$-modules corresponds to a map between their associated geometric vector bundles.

(2) (20 pts) Let $f : X \to Y$ be a map of schemes. Show that $\mathcal{L} \mapsto f^*\mathcal{L}$ induces a well defined homomorphism between their Picard groups

$$f^* : Pic(Y) \to Pic(X).$$

(3) (20 pts) Let X be a noetherian scheme and $\mathcal{F}$ be a coherent sheaf on X. Consider the function

$$\varphi(x) := dim_{k(x)} \left(\mathcal{F}_x \otimes_{\mathcal{O}_x} k(x)\right),$$

  where $k(x) := \mathcal{O}_x/\mathfrak{m}_x$ is the residue field at the point $x \in X$.
  - Show that the function $\varphi(x)$ is upper semicontinuous, i.e., for any $n \in \mathbb{Z}_{\geq 1}$, the set $\{x \in X \mid \varphi(x) \geq n\}$ is closed.
  - Show that if X is reduced and φ is a constant function, then $\mathcal{F}$ is locally free as an $\mathcal{O}_X$-module.

(4) (BONUS: 20 pts) Let k be an algebraically closed field with $char(k) \neq 2$ and $X \subset \mathbb{P}^2_k$ be the non-singular cubic curve given by

$$y^2 z = x^3 - xz^2.$$

  Let $Cl^0(X)$ denote the subgroup of the divisor class group of X that is defined to be the kernel of the degree map

$$Cl(X) \xrightarrow{deg} \mathbb{Z}.$$

  Let P_0 be the closed point $(0:1:0)$ on X. Let $X(k)$ be the set of all closed points of X. Define a map

$$\varphi : X(k) \to Cl^0(X)$$

  sending P to the divisor class $P - P_0 \in Cl^0(X)$. Show that this is a bijection. Hint: Any line L will meet X in exactly three closed points P, Q, R counting multiplicities. On the other hand, any two lines in $\mathbb{P}^2$ are linearly equivalent.

In particular, L is linearly equivalent to $\{z = 0\}$, whose divisor is $3P_0$. Therefore, $(P - P_0) + (Q - P_0) + (R - P_0) = 0$ in $Cl^0(X)$.