CHENNAI MATHEMATICAL INSTITUTE

SEMESTER II MIDTERM EXAMINATION 2022-2023

MU 2203– Topology (Undergraduate)

February 2023 TIME ALLOWED: 3 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **EIGHT (8)** questions and comprises **FOUR (4)** printed pages. **SEVEN (7)** questions out of these are mandatory, and **ONE (1)** question is for extra credit.

2. The total marks are out of 100, although it is possible to score more by solving the extra credit question. So if you solve that question, it is possible to improve your score, but not attempting it WILL NOT affect your grade.

3. This is a **closed book** examination. Books, notebooks, cellphones, laptops and any such objects that may enable you to get external help are not allowed.

4. You are expected to know all the terms and definitions appearing on the questions. If there is a definition that you are not expected to remember, it will be defined in the question.

5. Be brief but precise in your answers. You should justify your assertions.

Question 1

Recall that the minimal uncountable well-ordered set is denoted S_Ω. We will consider this as a topological space in its order topology.

(i) Show that a subset A of S_Ω is countable if and only if it is bounded as a subset of S_Ω. (5 marks)

(ii) Show that if A and B are two disjoint closed subsets of S_Ω, then at least one of them is countable and hence is bounded. (10 marks)

Total: 15 marks

Question 2

Let X be a topological space.

(i) Show that a countable topological space X (that is, X is countable as a set) is second countable if and only if it is first countable. (5 marks)

Appendix

(ii) Show that there exists a topological space X that is countable but not first countable by constructing an example. (10 marks)

Total: 15 marks

Question 3

This question is about the uniform metric defined on sets of the form $\mathbb{R}^J$ for J any set. Note that you are expected to know and remember its definition.

Let $\bar{\rho}$ be the uniform metric on $\mathbb{R}^\omega$. Given $x = (x_1, x_2, \ldots) \in \mathbb{R}^\omega$ and given $0 < \epsilon < 1$, let $U(x, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon) \times \cdots$.

(i) Show that $U(x, \epsilon)$ is not equal to the ϵ-ball $B_\epsilon(x)$ for the uniform metric $\bar{\rho}$. (5 marks)

(ii) Show that

$$B_\epsilon(x) = \bigcup_{\delta<\epsilon} U(x,\delta).$$

(10 marks)

Total: 15 marks

Question 4

Let X and Y be topological spaces. Let $p : X \to Y$ be a quotient map. Show that if X is locally connected, then Y is locally connected. (10 marks)

Total: 10 marks

Question 5

Let (X, d) be a metric space.

(i) If $f : X \to X$ is a function such that $d(f(x), f(x')) = d(x, x')$ for all $x, x' \in X$, then f is called an isometry of X. Show that if X is compact, and f is an isometry, then f is a homeomorphism. (12 marks)

(ii) Show that the compactness hypothesis is necessary above by constructing a counterexample. (3 marks)

Total: 15 marks

Question 6

Let $f : X \to Y$ be a continuous map and let X be a Hausdorff space and Y be a locally compact Hausdorff space. Then show that f is proper if and only if f is closed and for each $y \in Y$, the set $f^{-1}(y)$ is compact. (15 marks)

Total: 15 marks

Question 7

Let X be a topological space.

(i) If X is Hausdorff, show that the intersection of two compact subsets A and B of X is again compact. (5 marks)

(ii) Give an example of a non-Hausdorff space X with two compact subspaces A and B such that $A \cap B$ is not compact. (10 marks)

Total: 15 marks

Question 8

This is an **EXTRA CREDIT** question. The solution to this question may depend on *significantly more difficult or different concepts* than what you may have seen during this course. So it is NOT recommended that you attempt and spend time on this question before attempting other questions.

A game is played on $\mathbb{R}$, the set of real numbers, by two players. Player I begins by picking an uncountable subset X_1 of $\mathbb{R}$. Player II then picks an uncountable subset $X_2 \subseteq X_1$. (The game does not dictate that these have to be proper subsets.) The process then repeats and we have a sequence of uncountable subsets $\mathbb{R} \supseteq X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$. Define $X := \bigcap_n X_n$. Player I is declared to be the winner if X is nonempty, and Player II is declared to be the winner if $X = \varnothing$. Prove that no matter how Player I plays, Player II always has a winning strategy. (25 marks)

Total: 25 marks

END OF PAPER