CHENNAI MATHEMATICAL INSTITUTE

SEMESTER II ENDTERM EXAMINATION 2022-2023

MU 2203– Topology (Undergraduate)

April 2023 TIME ALLOWED: 3 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **NINE (9)** questions and comprises **FOUR (4)** printed pages. **SEVEN (7)** questions out of these are mandatory, and **TWO (2)** questions are for extra credit.

2. The total marks are out of 100, although it is possible to score more by solving the credit questions. So if you solve any of those questions, it is possible to improve your score, but not attempting them WILL NOT affect your grade.

3. This is a **closed book** examination. Books, notebooks, cellphones, laptops and any such objects that may enable you to get external help are not allowed.

4. You are expected to know all the terms and definitions appearing on the questions. If there is a definition that you are not expected to remember, it will be defined in the question.

5. Be brief but precise in your answers. You should justify your assertions.

Question 1

Let X and Y be topological spaces.

(i) Let X be a topological space that carries an order topology. Show that X is regular. (5 marks)

(ii) Let X and Y denote two topological spaces and let $f : X \to Y$ be a closed continuous surjective map. If X is a normal topological space, does it follow that Y is normal, too? Justify your assertion. (10 marks)

Total: 15 marks

Question 2

Let G be a topological group and H be a subgroup.

(i) If H is closed in G as a subspace, show that the quotient space G/H is a T_1 space. (5 marks)

(ii) If H is closed in G and H is a normal subgroup of G, then show that G/H is a topological group. (10 marks)

Total: 15 marks

Question 3

Let X be a nonempty set and $\mathcal{U}$ a filter on X. Then show that $\mathcal{U}$ is an ultrafilter if and only if for any subset $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$. (10 marks)

Total: 10 marks

Question 4

Let $p : E \to B$ be a covering map and let B be connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elements for every $b \in B$.

(15 marks)

Total: 15 marks

Question 5

Show that there does not exist a covering map from $\mathbb{RP}^2$ to the torus. (15 marks)

Total: 15 marks

Question 6

For spaces X and their subspaces A in the following examples, say whether A is a retract of X. As always, justify your assertions.

(i) X is the space obtained by taking two copies of the closed disc B^2 with a fixed basepoint b_0 on its boundary circle S^1 and identifying the two points b_0 together, i.e. gluing the two copies of B^2 together at the basepoint b_0; and A is the subspace that is its boundary, i.e. it is the figure eight space obtained by gluing the corresponding copies of S^1 together at the basepoint b_0. (10 marks)

(ii) X is the Möbius band; and A is the subspace that is its boundary circle. (10 marks)

Total: 20 marks

Question 7

Show that a space X is contractible if and only if X has the homotopy type of a one-point space.

(10 marks)

Total: 10 marks

Question 8

This is an **EXTRA CREDIT** question. The solution to this question may depend on *significantly more difficult or different concepts* than what you may have seen during this course. So it is NOT recommended that you attempt and spend time on this question before attempting other questions.

Let X and Y be topological spaces and $f : X \to Y$ a function between them with the following property : For any M a metric space and any continuous map $g : M \to X$, the composite $f \circ g : M \to Y$ is a continuous map, too. Does it follow that f is a continuous map? (20 marks)

Total: 20 marks

Question 9

This is an **EXTRA CREDIT** question. The solution to this question may depend on *significantly more difficult or different concepts* than what you may have seen during this course. So it is NOT recommended that you attempt and spend time on this question before attempting other questions.

Prove or disprove the following statement : The Cantor set can be given the structure of a topological group. (20 marks)

Total: 20 marks

END OF PAPER