CHENNAI MATHEMATICAL INSTITUTE

Geometric complexity theory

Mid Term examination:
Date: Feb 27, 2023. 3:30 pm - 6:30 pm

(1) Show that the image of the morphism $\mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \mapsto (x, xy)$ can be written as the union of two algebraic subsets. Describe those and write down their coordinate rings. Is the image an algebraic subset of $\mathbb{A}^2$? Give reasons. (5 marks)

(2) Recall that a map of rings $\phi : R \to R'$ is said to be integral if each element in R' satisfies a monic polynomial with coefficients in R. Now suppose A, B are integral domains and $A \to B$ is an injective integral ring homomorphism. Show that A is a field if and only if B is a field. 5 marks

(3) Consider the set of 2x2 matrices of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ where $a \in \mathbb{C}$. Show that this is an algebraic group and describe its coordinate ring? The action of G on $\mathbb{C}^2$ gives an action of G on the coordinate ring $\mathbb{C}[x, y]$. Write down this action. Consider the induced action on the ring $k[x, y]/(x^2)$. Show that the invariant ring is generated by xy^n, $n \geq 0$. Prove or disprove: the ring of invariants is finitely generated. 10 marks

(4) Let G be an algebraic group acting on an algebraic subset V. Show that there is a linear representation W of G and a G-equivariant embedding of V in W. Show that G is isomorphic to a closed subgroup of $GL(n)$. for some n, i.e. it is a linear algebraic group. You may proceed by taking a set of generators for $k[V]$. 5 marks.

Now let H be a closed subgroup of G, closed in the Zariski topology. Let I be the ideal defining H. Show that H acts on I. Show that we can select a H-stable subspace W of I which generates I as an ideal. Show that there is a G-stable subspace U of $k[G]$ containing W. For the action of G on U, show that H is precisely those elements of G which send W back to itself, i.e. $H = \{g \in G | gW = W\}$. 5 marks.

(5) Consider the map from $\mathbb{P}^1$ to $\mathbb{P}^3$ given by $[s : t] \to [s^3, s^2t, st^2, t^3]$. Is this a well defined map? Why? Assume the coordinates on $\mathbb{P}^3$ are $[X : Y : Z : W]$. Show that this is a bijection onto the image and compute the inverse. Show that the image is the curve C given by the vanishing of 2x2 determinants of the matrix $\begin{pmatrix} X & Y & Z \\ Y & Z & W \end{pmatrix}$. Show that the intersection of any two of the 3 equations is the union of C and a line. You may do this for one choice of two of these equations. 5 marks

(6) Let f be a homogeneous polynomial of degree d in $\mathbb{C}[x_1, x_2, \ldots, x_n]$. Let $m < d$ and consider the set S of all monomials of degree m in the variables $x_1, \ldots, x_n$. For each element $s = x_{i_1}x_{i_2}\cdots x_{i_m} \in S$ (repetitions allowed) consider the polynomial of degree $\leq d - m$, $\frac{\partial^m f}{\partial x_{i_1} \ldots \partial x_{i_m}}$, obtained from f and express this as a vector ∂f_s of coefficients in the basis of monomials of degree $d-m$. Now construct the matrix ∂f of size $\ell \times |S|$ with the s-th column being the vector ∂f_s - ℓ being the dimension of monomials of degree $d - m$. What is the rank of the matrix when $f = x_1^d$? Show that the border Waring rank of a polynomial f is at least as large as the rank of the matrix ∂f 5 marks.

(7) Let G be an algebraic group and V an irreducible subvariety of G with $1 \in V$. Let V^{-1} be the set of inverses of V. Show that $U := \overline{VV^{-1}}$ is an irreducible algebraic variety. For every positive integer m, let $\overline{U^m}$ be the closure of $UU \cdots U$ (taken m times). Show that $\overline{U^m}$ is an irreducible closed variety. Use these observations to show that the closed subgroup H of G generated by V is an algebraic group. 5 marks