CHENNAI MATHEMATICAL INSTITUTE

Geometric complexity theory

End Term examination:
Date: April 29, 2023.

Answer any five of the first six questions (each is 8 marks), any one of question 7 or 8 (each is 12 marks). All questions have to be attempted when submitting the take home version. Question 9 carries 18 marks. The final weightage is 40 marks.

(1) Let X be a $GL(n)$ variety and S be a closed $GL(n)$-subvariety of X. Show that there is a $GL(n)$ representation W and a $GL(n)$-morphism $\phi : X \to W$ such that S is the inverse image of the zero vector in W.

(2) Let G be an algebraic group acting on an algebraic subset V. Suppose $v \in V$ is such its orbit has minimal dimension among orbits in V. Prove that $G \cdot v$ is closed.

(3) Consider the action of $GL(3)$ on $V^{\otimes 3}$, V being the standard representation of $GL(3)$. Show that the vector $2e_1 \otimes e_1 \otimes e_2 - e_2 \otimes e_1 \otimes e_1 - e_1 \otimes e_2 \otimes e_1$ is a highest weight vector. Show that the dimension of the $GL(3)$-module containing this vector is more than one.

(4) Let H be the subgroup of $GL(2, \mathbb{C})$ generated by matrices of the form $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Is H closed in $GL(2, \mathbb{C})$?

(5) Let T be a torus acting on a variety X. Show that the ring of invariants is generated by monomials. Let T be the torus of diagonal matrices in $SL(3)$. Consider the action of $T \times T$ on 3 by 3 matrices given by $(l_1, r_1) \cdot M = t_1 M r_1^{-1}$. What are the generators of the ring of invariants for this action? You may assume that the ring of invariants is generated by monomials and give a sketch as to why your generators will generate the entire ring of invariants. Give a combinatorial description for the null cone for this action? Recall that the null cone is the zero set of all non-constant homogeneous invariant functions.

(6) Let V be a representation of $GL(n)$ and suppose v is a weight vector of weight λ. Assume you have fixed a basis of weight vectors for V. Consider a unipotent, lower triangular element g having nonzero entries below the diagonal, except for one $i > j$, where $g(i, j) = r$. Write down the expression for the action of g on v as a formal linear combination of weight vectors, with coefficients being polynomials in r. If $\lambda := (\lambda_1, \ldots, \lambda_n)$ is a partition, write down the n-tuples representing the weights of vectors in the support of gv in terms of λ, i, j.

(7) Recall that an affine algebraic set is specified by a pair (V, A) where V is a set and A is a $\mathbb{C}$ algebra of functions on V with values in $\mathbb{C}$ with the property that A is finitely generated over $\mathbb{C}$, A separates points, so given $x, y \in V$, there is an $f \in A$, $f(x) \neq f(y)$. Furthermore every $\mathbb{C}$-algebra homomorphism from A to $\mathbb{C}$ is given by evaluation at a point in V, so for every $\phi : A \to \mathbb{C}$, there is an $x \in V$ such that for all $f \in A$, $\phi(f) = f(x)$, so $\phi = e_x$ the evaluation at x.

Assume the following lemma:

Lemma Let R and S be integral domains with $S \subseteq R$ and R finitely generated over S. Let $f \neq 0, f \in R$. Then there exists a $g \neq 0, g \in S$ such that for any algebraically closed field F and a homomorphism $\phi : S \to F$ with $\phi(g) \neq 0$, ϕ can be extended to $\tilde{\phi} : R \to F$ such that $\tilde{\phi}(f) \neq 0$.

Assume we have a morphism between algebraic subsets $\alpha : U \to V$. Suppose there exists $U' \subseteq U$ such that U' is irreducible and U' contains a dense open set of its closure. Show that $\alpha(U')$ also has the same properties - it is irreducible and contains a dense open set of its closure.

Proceed in the following steps. Show first that the the given U can be replaced by $\overline{U'}$, so we may assume that U' is dense in U. Argue that U' can assumed to be a principal open set. Argue like before that we may assume that $\alpha(U)$ is dense in V by replacing V appropriately. Then conclude.

(8) Let V be the representation of a reductive group G. Let f be a homogeneous invariant polynomial of degree d. Given parameters t_1, t_2 write

$$f(t_1 v_1 + t_2 v_2) = \sum_{i_1, i_2} P_{i_1,i_2} f(v_1, v_2) t_1^{i_1} t_2^{i_2}$$

the sum running over all i_1, i_2 such that $i_1 + i_2 = d$. The polynomials $P_{i_1,i_2} f(v_1, v_2)$ can be interpreted as regular functions on $V^{\oplus 2}$ of 2, the vector space of the direct sum of two copies of V with the diagonal action of G, i.e. $g(w_1, w_2) = (g(w_1), g(w_2))$. Show that the polynomials $P_{i_1,i_2} f(v_1, v_2)$ are invariant for this diagonal action of G on two copies of V. Work this out for the representation of $\mathbb{C}^*$ on polynomials on $\mathbb{C}^2$ given by $t(x, y) = (tx, t^{-1}y)$, when we take two copies of on $\mathbb{C}^2$. Do these polynomials generate the ring of invariants for the diagonal action of $\mathbb{C}^*$ on two copies of $\mathbb{C}^2$?

(9) Take a degree 2 form in three variables x, y, z and denote the coefficients of $x^2, y^2, z^2, xy, xz, yz$ as $T_1, T_2, T_3, T_4, T_5, T_6$, respectively. Consider the action of $GL(3)$ on these coefficients. Write down the action of a generic element on each of $T_1, T_2, \ldots, T_6$. Show that the polynomial $4T_1T_2T_3 + T_4T_5T_6 - T_1T_6^2 - T_2T_5^2 - T_3T_4^2$ is a highest weight vectors for $S^3S^2(C^3)$ under the identification of $S^2(C^3)$ with

a generic polynomial of degree two in x, y, z. What is the dimension of this module?