DEQN
Instructor: Siddhi Pathak
February 24, 2023

Mid-term Exam

Max. Marks: 80
Duration: 3 hours.

Please write complete proofs if you wish for full marks! You are allowed to use the theorems we have proved in class. But make sure to refer to the theorem whenever you use it.

1. (12 points) Let $D \subseteq \mathbb{R} \times \mathbb{R}$ be open and $(t_0, x_0) \in D$ such that $[t_0 - \alpha, t_0 + \alpha] \times \overline{B(x_0, r)} \subseteq D$ for some $r > 0$. Let $v_1, v_2 : D \to \mathbb{R}$ be continuous vector fields on D, bounded by M on $[t_0 - \alpha, t_0 + \alpha] \times \overline{B(x_0, r)}$. Set $b := \min(\alpha, r/M)$. For $j = 1$ and 2, let $\phi_j : [t_0 - b, t_0 + b] \to \mathbb{R}$ be any solution of the initial value problems $\dot{x} = v_j(t, x)$ and $x(t_0) = x_0$ (guaranteed to exist by Cauchy-Peano).

   (a) (7 points) Suppose that
   $$v_1(t, x) > v_2(t, x)$$
   for all $(t, x) \in D$. Then prove that
   $$\phi_1(t) \leq \phi_2(t) \ \forall \ t_0 - b < t \leq t_0, \qquad \text{and} \qquad \phi_1(t) \geq \phi_2(t) \ \forall \ t_0 \leq t < t_0 + b \tag{1}$$

   (b) (5 points) Suppose that
   $$v_1(t, x) \geq v_2(t, x)$$
   for all $(t, x) \in D$. Is the above conclusion in (1) still true? If yes, prove it. If no, provide a counterexample.

2. (13 points) Let D be an open interval in $\mathbb{R}$ and $v : D \to \mathbb{R}$ be a continuously differentiable (autonomous) vector field on D.

   (a) (3 points) Show that v is locally Lipschitz continuous with respect to x on D.

   (b) (5 points) Let $\phi_0 : (\omega_-, \omega_+) \to \mathbb{R}$ denote the maximal solution of the IVP $\dot{x} = v(x)$ and $x(t_0) = x_0$ for some $x_0 \in D$. State whether the following statement is True or False. Justify your answer.
   The limit
   $$\lim_{t \to \omega_+} \phi_0(t) < \infty$$
   always exists in $\mathbb{R}$.

   (c) (5 points) Given that $\lim_{t \to \omega_+} \phi_0(t) = L$ exists and $L \in D$. Show that $\omega_+ = +\infty$ and hence, $v(L) = 0$. write clearly.

3. (7 points) Let $U \subseteq \mathbb{R} \times \mathbb{R}$ be open and $v(t, x)$ be locally Lipschitz with respect to x on U. For some $(t_0, x_0) \in U$, let $\phi_0 : (\lambda_-, \lambda_+) \to \mathbb{R}$ denote the maximal solution of the IVP $\dot{x} = v(t, x)$ and $x(t_0) = x_0$. State whether the following statement is True or False. Justify your answer.
   Suppose that $|\phi_0(t)|$ is bounded as t tends to λ_+. Then the limit
   $$\lim_{t \to \lambda_+} \phi_0(t)$$
   always exists in $\mathbb{R}$.

4. (8 points) Consider the vector field $v(x) := 2\sqrt{|x|}$ on $\mathbb{R}$.

   (a) (3 points) How many distinct solutions, defined on $\mathbb{R}$, does the IVP $\dot{x} = v(x)$ and $x(0) = 0$ have? Write down a formula for each of them. (You need not *prove* that your list of solutions is exhaustive.)

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(b) (5 points) How many distinct solutions, defined on $\mathbb{R}$, does the IVP $\dot{x} = v(x)$ and $x(2) = 4$ have? (You need not *prove* that your list of solutions is exhaustive.) Write down a formula for each of them. Does your answer contradict the local existence and uniqueness theorem? Explain.

5. (10 points) In each of the instances below, provide an example, with justification, of a continuous vector field, $v : \mathbb{R} \to \mathbb{R}$ satisfying the given conditions.

(a) (5 points)

- there exists a point $x_0 \in \mathbb{R}$ such that $v(x_0) = 0$ and $v(x)$ is not differentiable at x_0.
- the IVP $\dot{x} = v(x)$ and $x(0) = x_0$ has a unique solution defined on $\mathbb{R}$.

? (b) (5 points)

- there exists a point $x_0 \in \mathbb{R}$ such that $v(x)$ is NOT Lipschitz continuous on any neighborhood of x_0.
- the IVP $\dot{x} = v(x)$ and $x(0) = x_0$ has a unique solution in a neighborhood of $t = 0$.

6. (15 points) Picard iterates - Recall the following.

- Let $U \subseteq \mathbb{R}^n$ be a domain and $x_0 \in U$ with $\overline{B(x_0, r)} \subseteq U$. Let $v(t, x) : [t_0 - \alpha, t_0 + \alpha] \times U \to \mathbb{R}^n$ be a continuous vector field, bounded by M on $[t_0 - \alpha, t_0 + \alpha] \times \overline{B(x_0, r)}$ and Lipschitz with respect to x on $[t_0 - \alpha, t_0 + \alpha] \times U$ with Lipschitz constant L.
- Let $b := \min\{\alpha, r/M\}$. Let $\widetilde{M} := \max\{|v(t, x_0)| : t \in [t_0 - \alpha, t_0 + \alpha]\}$.
- Let $\mathcal{X}$ be the set of continuous maps from $I := [t_0 - b, t_0 + b]$ to $\overline{B(x_0, r)}$. Define $T : \mathcal{X} \to \mathcal{X}$ by $\phi \mapsto T(\phi)$ with

$$T(\phi)(t) := x_0 + \int_{t_0}^{t} v(s, \phi(s))\, ds.$$

- Define a sequence of functions, $\phi_n : I \to \mathbb{R}^n$ by

$$\phi_0 \equiv x_0,\ \phi_1 = T(\phi_0),\ \phi_2 = T(\phi_1) = T^2(\phi_0), \cdots, \phi_n = T^n(\phi_0), \cdots$$

These sequence of functions are called Picard approximations or Picard iterates.

(a) (8 points) Show that for each n, the sequence of functions, $\{\phi_n\}_{n \geq 0}$ converge uniformly to a continuous function ϕ on I. (You do need not prove the continuity of ϕ, but deduce it by invoking a standard theorem. State this theorem!)

(b) (7 points) Show that for each $n > 0$ and $t \in I$,

$$|\phi(t) - \phi_n(t)| \leq \frac{\widetilde{M}}{L} \frac{(Lb)^{n+1}}{(n+1)!} e^{Lb}.$$

7. (8 points) Let $(\mathbb{R}, \{g^t\})$ be a 1-parameter group of linear transformations of $\mathbb{R}$. Then show that for all $x, t \in \mathbb{R}$, $g^t(x) = x\, e^{kt}$ for some $k \in \mathbb{R}$.

8. (7 points) Find the general solution of the following linear system -

$$\dot{x}_1 = x_1 - x_2$$
$$\dot{x}_2 = x_2$$