DEQN
Instructor: Siddhi Pathak
April 25, 2023

Final Exam

Total Marks: 80
Duration: 3 hours

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1. (7 points) State whether the following statement is true or false. If true, provide a proof. If false, provide a counterexample. You can also use the fact: if $\psi \in C^1([\alpha, \beta])$ is such that $\psi'(t) \leq K\psi(t)$ for $t \in [\alpha, \beta]$ and $K \geq 0$, then $\psi(t) \leq \psi(\alpha)\, e^{K(t-\alpha)}$ for $t \in [\alpha, \beta]$.

   Let $\mathcal{D} \subseteq \mathbb{R} \times \mathbb{R}$ be open and $(t_0, x_0) \in \mathcal{D}$ such that $[t_0 - \alpha, t_0 + \alpha] \times \overline{B(x_0, r)} \subseteq \mathcal{D}$ for some $r > 0$. Let $v_1, v_2 : \mathcal{D} \to \mathbb{R}$ be continuous vector fields on $\mathcal{D}$, bounded by M on $[t_0 - \alpha, t_0 + \alpha] \times \overline{B(x_0, r)}$. Set $b := \min(\alpha, r/M)$. For $j = 1$ and 2, let $\phi_j : [t_0 - b, t_0 + b] \to \mathbb{R}$ be any solution of the initial value problems $\dot{x} = v_j(t, x)$ and $x(t_0) = x_0$ (guaranteed to exist by Cauchy-Peano). Suppose that

$$v_1(t, x) \geq v_2(t, x)$$

   for all $(t, x) \in \mathcal{D}$. Further assume that v_1 is Lipschitz continuous with respect to phase in $\mathcal{D}$, that is, there exists a constant $L > 0$ such that for all (t, x_1) and $(t, x_2) \in \mathcal{D}$,

$$|v_1(t, x_1) - v_1(t, x_2)| \leq L|x_1 - x_2|. \qquad \text{Then}$$

$$\phi_1(t) \leq \phi_2(t)\ \forall\ t_0 - b < t \leq t_0, \qquad \text{and} \qquad \phi_1(t) \geq \phi_2(t)\ \forall\ t_0 \leq t < t_0 + b. \tag{1}$$

2. (5 points) Show that there is a one parameter group of diffeomorphisms on $(-\pi/2, \pi/2)$ whose phase velocity is $v(x) = \cos(x)$.

3. (5 points) Solve the following initial value problem -

$$x^{(3)} - x^{(2)} + 4x^{(1)} - 4x = 0, \qquad x(0) = 5,\ x'(0) = 8,\ x''(0) = -12.$$

? 4. (5 points) Let $(b_n)_{n \geq 0}$ be a sequence of rational numbers defined by the recurrence relation

$$b_{n+3} = b_{n+2} - 4b_{n+1} + 4b_n \text{ for } n \geq 0, \qquad \text{with } b_0 = 5,\ b_1 = 8,\ b_2 = -12.$$

   Find b_{2023}.

5. (23 points) State whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

   (a) (5 points) Let $A : \mathbb{R} \to M_n(\mathbb{R})$ be a continuous function, periodic with period $\omega > 0$. Then the IVP $\dot{\vec{x}} = A(t)\vec{x}$ and $\vec{x}(0) = \vec{x}_0$ has a periodic solution.

   ? (b) (8 points) There exists a function $w(x) \in C^1(\mathbb{R})$ such that the phase portrait of $\dot{x} = w(x)$ is same as that of $\dot{x} = x^2$, but the solutions of the IVP

$$\dot{x} = w(x) \qquad x(0) = x_0$$

   exist for all time $t \in (-\infty, \infty)$, for all $x_0 \in \mathbb{R}$.

   (c) (5 points) The map $e^X : M_n(\mathbb{R}) \to GL_n^+(\mathbb{R})$ given by $A \mapsto e^A$ is surjective. Here

$$GL_n^+(\mathbb{R}) := \{A \in GL_n(\mathbb{R}) \ :\ \det A > 0\}.$$

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(d) (5 points) The map $e^X : M_n(\mathbb{C}) \to GL_n(\mathbb{C})$ given by $A \mapsto e^A$ is surjective. Here

$$GL_n(\mathbb{C}) := \{A \in M_n(\mathbb{C}) \,:\, \det A \neq 0\}\,.$$

6. (15 points) Consider the second order linear differential equation

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0, \tag{2}$$

where $p(t)$ and $q(t)$ are continuous functions on $\mathbb{R}$. Let $f(t)$ and $g(t)$ be two linearly independent solutions of (2). Let t_1 and $t_2 \in \mathbb{R}$ be consecutive zeros of $g(t)$ (that is, $g(t_1) = g(t_2) = 0$ and $g(t) \neq 0$ on (t_1, t_2)). Then

(a) (5 points) Prove that the function $f(t)$ is non-zero at $t = t_1$ and $t = t_2$.

(b) (10 points) Prove that there exists a $t_0 \in (t_1, t_2)$ such that $f(t_0) = 0$.

7. (20 points) For the following systems of differential equations, determine whether the origin is a stable, asymptotically stable or unstable equilibrium point. Justify your conclusion by naming/stating the relevant theorems from the course and proper computations.

9. (a) (10 points)

$$\begin{aligned}
\dot{x_1} &= -x_1 + 4x_1x_2 + 2x_3 + x_4^2 \\
\dot{x_2} &= 2x_1^2 + 3x_2 - 4x_3 - 2x_4 \\
\dot{x_3} &= x_1x_2 + 3x_1x_2x_3 - x_3 - 2x_4 \\
\dot{x_4} &= 2x_1x_4 + 2x_3 - x_4
\end{aligned}$$

(b) (10 points)

$$\begin{aligned}
\dot{x_1} &= 3x_1 \\
\dot{x_2} &= x_1x_3 - 2x_3 \\
\dot{x_3} &= 2x_2 - x_1x_2
\end{aligned}$$

8. (**BONUS** Problem - 20 points) In Q.1, instead of the Lipschitz condition, assume that the IVP's $\dot{x} = v_1(t, x)$, $x(t_0) = x_0$ and $\dot{x} = v_2(t, x)$, $x(t_0) = x_0$ have unique solutions in $[t_0 - b, t_0 + b]$ for all $(t_0, x_0) \in \mathcal{D}$. Then does (1) hold? If yes, give a proof. If no, give a counterexample.