Complex Analysis
Mid-semester exam
23/04/23, 9:30am-12:30pm

You may use your class notes and textbook. No other sources are permitted. In a multi-part question, the result of one part may be used in a succeeding part even if you have not been able to derive it.

1. (*5+5=10 points*) Compute the following line integrals.

(a)

$$\int_{|z|=1} \frac{e^z dz}{z}$$

(b)

$$\int_{|z|=2} \frac{dz}{z^2+1}$$

2. (*5+5=10 points*) Determine the values of z for which the following series are convergent.

(a)

$$\sum_{n=0}^{\infty} \left(\frac{z}{1+z}\right)^n$$ Appendix.

(b)

$$\sum_{n=0}^{\infty} \frac{z^n}{1+z^{2n}}$$ Appendix

appendix 3. (*10 points*) Map the open region between $|z| = 1$ and $|z - \frac{1}{2}| = \frac{1}{2}$ conformally onto a half plane.

4. (*5+5=10 points*) Let $\Omega \subseteq \mathbb{C}$ be a region and $\gamma \subset \Omega$ any closed curve. Suppose that $f(z)$ is analytic on Ω.

(a) Show that

$$\int_\gamma \overline{f(z)} f'(z) dz$$

is purely imaginary.

(b) If $f(z)$ satisfies the inequality $|f(z) - 1| < 1$ on Ω, show that

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 0.$$

5. (*10 points*) Let $\Omega \subset \mathbb{C}$ be an open region, and $\{f_n\}_{n=1}^{\infty}$ a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω. Show that f is holomorphic in Ω. Is the same result true of a sequence of C^∞ functions: if $\{f_n\}_{n=1}^{\infty}$ a sequence of smooth (real) functions that converges uniformly to a function f, then is f smooth as well?

(write properly).

6. (*10 points*) Let f be analytic on the open unit disc D. Show that the diameter $d := \sup_{z,w \in D} |f(z) - f(w)|$ of the image of f satisfies:

Appen lic

$$2|f'(0)| \leq d.$$

Show that equality holds iff f is linear, that is, $f(z) = a_0 + a_1 z$ for some $a_0, a_1 \in \mathbb{C}$. (Hint: $2f'(0) = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z) - f(-z)}{z^2} dz$ for any $\rho < 1$).

7. Let $\xi \in \mathbb{R}_{\geq 0}$ be a constant. Show that

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i \xi x} dx = e^{-\pi \xi^2}$$

in the following steps. For $z \in \mathbb{C}$, let $f(z) = e^{-\pi z^2}$; for $R > 0$, let γ_R be the counterclockwise oriented rectangle in the plane with vertices $-R, R, R + i\xi$ and $R - i\xi$. Then,

$$\int_{\gamma_R} f dz = \int_1 f dz + \int_2 f dz + \int_3 f dz + \int_4 f dz,$$

where $\int_1$ is the integral along the path from $-R$ to R, $\int_2$ that from R to $R + i\xi$, and so on.

(a) Show that $\lim_{R\to\infty} \int_2 f dz = 0$. Similarly, $\lim_{R\to\infty} \int_4 f dz = 0$.

(b) Show that $\int_3 f dz = -e^{\pi\xi^2} \int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i \xi x} dx$.

(c) Assume that $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$. Complete the calculation using Cauchy's Theorem or otherwise.

The same calculation works for $\xi < 0$ too.