# CANA Final Exam

09:30-12:30, 24 April, 2023

**You may use your class notes or the textbook during the exam; no other sources are permitted. When attempting a multi-part problem, you may assume the result of one part in a subsequent part without deriving it. Each problem is worth 10 points.**

1. Compute the following integral by the method of residues (or otherwise):
$$\int_0^\infty \frac{\log(1+x^2)}{x^{1+\alpha}}dx, \text{ where } 0 < \alpha < 2.$$

2. Let $\{f_n\}$ be a sequence of analytic function defined on a region $\Omega$. Suppose that $\{f_n\}$ converges to a function $f(z)$, uniformly on every compact subset of $\Omega$. If the $f_n(z)$ have at most $m$ zeros in $\Omega$, show that $f(z)$ is either identically zero or has at most $m$ zeros.

3. Prove that any entire function $f$ that is also injective must be of the form $f(z) = az + b$, with $a, b \in \mathbb{C}$, $a \neq 0$. (Hint: Recall the theorem of Weierstrass that an analytic function comes arbitrarily close to any complex value in every neighbourhood of an essential singularity. Apply this to $f(1/z)$.)

4. Let $f(z)$ be an analytic function on the open unit disc D such that that $|f(z)| < 1$ for all $z \in D$. Suppose that there exist two distinct points $a, b \in D$ that are fixed by $f$, that is, $f(a) = a$, $f(b) = b$. Show that $f(z) = z$ for all $z \in D$ as follows:

   (a) Let be the linear fractional transformation
$$\phi_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}.$$

Show that if $\alpha \in D$, then $\phi_\alpha$ maps $D$ onto itself bijectively.

(b) Apply Schwarz's Lemma to $\phi_a \circ f \circ \phi_{-a}$ and conclude.

5. Evaluate the integral

$$\int_\gamma e^{e^{1/z}}\,dz$$

where $\gamma$ is the circle of radius 2 centred at the origin. (Hint: Series expansions for the exponential function.)

6. Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be analytic with a pole of order 1 at 0. If $f(z) \in \mathbb{R}$ for all $|z| = 1$, show that for some nonzero $\alpha \in \mathbb{C}$, and $\beta \in \mathbb{R}$,

$$f(z) = \alpha z + \overline{\alpha}\frac{1}{z} + \beta, \quad \forall z \neq 0.$$

(Hint: Determine the coefficients of the series expansion of $f$.)

7. Show that the following series defines a meromorphic function on $\mathbb{C}$, and determine the set of poles and their orders.

$$f(z) = \sum_{n=0}^{\infty} \frac{\sin(nz)}{n!(z^2 + n^2)}.$$