Algebraic Geometry II, Spring 2023
Mid-sem exam
12 April, 2023

<u>Notation</u>:
– k always denotes an algebraically closed field.
– A variety over k is an integral separated scheme of finite type over k.
– A *curve* is a nonsingular, projective variety over k of dimension 1.
– A point always refers to a closed point.

1. Let X be a curve of genus g and let D be a effective divisor on X. Show that

$$\dim |D| \leqslant \deg D,$$

and that the equality holds if and only of $D = 0$ or $g = 0$.

2. Let X be a curve and let $P \in X$. Show that there exists a nonconstant rational function $f \in K(X)$, which is regular everywhere except at P.

   Hint: Consider sections of the line bundle $\mathcal{O}_X(np)$ for large integers n.

3. Let X be a curve. The *gonality* of X is defined to be the least integer d such that X admits a finite morphism $X \to \mathbb{P}^1$ of degree d.

   (a) Suppose that X is a smooth plane curve of degree d. Show that the gonality of X is at most $d - 1$.

   (It is a classical theorem of M. Noether that the gonality is actually equal to $d - 1$ when $d \geqslant 2$.)

   (b) Let X be a curve of genus g. Show that the gonality of X is at most $g + 1$.

4. Show that there is no curve of degree 9 and genus 11 in $\mathbb{P}^3$.

   Hint: If such a curve exists then show that it must lie on a quadric surface. For this, you may need to use Clifford's theorem.

   To conclude the proof, use the following facts (without proof):

   - If X is a curve of type (a, b) on a nonsingular quadric surface, then the degree of X is $a + b$ and the genus is $ab - a - b + 1$.
   - If X is a curve of odd degree $d = 2a + 1$ on a quadric cone in $\mathbb{P}^3$, then the genus of X is $a^2 - a$.