Notation :

Unless otherwise stated, the following is the default notation.

$R, S, A, B, \cdots$: commutative rings

$I, J, \mathfrak{a}, \mathfrak{b}, \mathfrak{p}, \mathfrak{q}$: ideals.

$\Bbbk, \ell$: fields

$M, N, \cdots$: modules.

Date and Time : 2022-Sep-25 14:15-17:15

Answer all the questions

Total marks: 165

Contribution towards final grade: (your marks)/5 or 30 whichever is lower.

1. True / False. Give sufficient justification for your answers.
   (a) (5 points) If R is a noetherian ring with $|\operatorname{Spec} R| < \infty$, then R is artinian.
   (b) (5 points) $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is an integral domain.
   (c) (5 points) $\Bbbk[x] \otimes_{\Bbbk} \Bbbk[x] \simeq \Bbbk[x, y]$.

2. (10 points) Let $R = \mathbb{Q}[u, v, x, y]$ and $I = (ux, vy, uy + vx)$. Determine $\operatorname{Ass}(R/I)$.

3. Let $\phi : \mathbb{C}[x, y] \to \mathbb{C}[s, t], x \mapsto s^2, y \mapsto st$. Let $\psi : \mathbb{C}^2 \to \mathbb{C}^2$ be the corresponding map. Prove or disprove the following statements.
   (a) (5 points) ψ is surjective.
   (b) (5 points) $\operatorname{Im}(\psi)$ is dense in the Zariski topology on $\mathbb{C}^2$.
   (c) (5 points) There exist $p \in \operatorname{Im}(\psi)$ such that $\psi^{-1}(p)$ is a finite set.
   (d) (5 points) There exist $p \in \operatorname{Im}(\psi)$ such that $\psi^{-1}(p)$ is an infinite set.

4. (a) (10 points) Let $(R, \mathfrak{m})$ be a local ring and M, N finitely generated non-zero R-modules. Show that $M \otimes_R N \neq 0$.
   (b) (10 points) Let R be a ring and M, N finitely generated R-modules. Then $M \otimes_R N = 0$ if and only if $\operatorname{Supp}(M) \cap \operatorname{Supp}(N) = \varnothing$.
   (Hint: $U^{-1}(M \otimes_R N) \simeq U^{-1}M \otimes_{U^{-1}R} U^{-1}N$ for each multiplicatively closed $U \subseteq R$.)

5. (a) (5 points) Show that if R is a noetherian reduced ring, then $\operatorname{Min}(R) = \operatorname{Ass}(R)$.
   (b) (5 points) Give an example of a non-reduced noetherian ring of dimension 1 for which $\operatorname{Min}(R) = \operatorname{Ass}(R)$.
   (c) (10 points) Show that a ring R is artinian and reduced if and only if it is a product of a finite collection of fields.

6. Assume that R is noetherian. Let I be an R-ideal and $a \in R$. Let $I = J_1 \cap J_2 \cap \cdots \cap J_m$ be a minimal irredundant primary decomposition of I. Assume that $a \notin \sqrt{J_i}$ for all $1 \leq i \leq l$ and $a \in \sqrt{J_i}$ for all $l + 1 \leq i \leq m$. Write $(I : a^\infty) = \cup_{e \geq 1}(I : a^e)$. This is called the *saturation* of I by a.
   (a) (5 points) There exists $e \geq 1$ such that $(I : a^\infty) = (I : a^e)$.
   (b) (10 points) $(I : a^\infty) = (J_1 : a^\infty) \cap (J_2 : a^\infty) \cap \cdots \cap (J_m : a^\infty)$
   (c) (10 points)
   $$J_i : a^\infty = \begin{cases} J_i, & 1 \leq i \leq l; \\ R, & r + 1 \leq i \leq m. \end{cases}$$
   (d) (10 points) Determine $\operatorname{Ass}(R/(I : a^\infty))$.

7. (10 points) Let $R \subseteq S$ be rings and $s \in S$. Say that s is *integral* over R if there exist $r_1, \ldots, r_n$ such that $s^n + \sum_{i=1}^{n} r_i s^{n-i} = 0$. Let R be a UFD that is not a field and S its field of fractions. Show that no element of $S \smallsetminus R$ is integral over R.

8. Say that an R-module M is *simple* if $M \neq 0$ and there does not exist a submodule N of M such that $0 \neq N \subsetneq M$.

   (a) (5 points) Let M be a simple R-module. Show that there exists a maximal ideal $\mathfrak{m}$ of R such that $M \simeq R/\mathfrak{m}$.

   (b) (10 points) Let M be a finitely generated R-module. Show that then there exists a submodule N such that M/N is simple.

   (c) (10 points) Using the above step, prove the following: If J is the Jacobson radical of R and M is a non-zero finitely generated R-module, then $JM \neq M$.

9. (10 points) Let $(R, \mathfrak{m})$ be a noetherian local ring and M a finitely generated R-module. Assume that $\mathfrak{m} \notin \operatorname{Ass}(R) \cup \operatorname{Ass}(M)$. Show that there exists $l \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ that is a nzd both on R and on M.