CMI Calculus 2022 (MU2103) Midsem

09:30-12:30, Thursday, 29 September 2022

Write clearly at the start of the answer book: *Calculus Midsem. Your name and roll number.* Number the pages. You have to justify your steps. Credit for neatness and brevity.

When in doubt, try the mean-value theorem. As usual, $I = [0, 1]$.

Part A: `Maximum Marks 20/25 ±1`[1]

(1) [3] Consider

$$T = \{(x, y, 0) | 0 < x < 1,\ 0 < y < 1,\ x \textit{ and } y \textit{ rational}\} \subset \mathbb{R}^3$$

What is the boundary of T? Let χ_T denote the indicator function of T. Which are the points of continuity of χ_T? What is $\chi_T((\frac{\sqrt{2}}{2}, \frac{1}{2}, 0))$?

(2) [3] Consider the function $h : \mathbb{R}^2 \to \mathbb{R}^1$, defined by

$$h(x, y) = \begin{cases} 0 & if\ r \leq 1 \\ r - 1 & if\ 1 < r \leq 2 \\ 3 - r & if\ 2 < r \leq 3 \\ 0 & if\ 3 > r \end{cases}$$

where $r = \sqrt{x^2 + y^2}$ (positive square root). Draw a graph of h (as best as you can) as a surface in $\mathbb{R}^3$. What is the support of h? Is this a function of compact support?

(3) [3] Let P_1 be the partition $\{[0, 1/2], [1/2, 1]\}$ of I and P_2 the partition $\{[0, 1/3], [1/3, 2/3], [2/3, 1]\}$. Consider the two product partitions of the square $I \times I$

$$Q_1 = P_1 \times P_2, \quad and \quad Q_2 = P_2 \times P_1$$

Find a common refinement Q of Q_1 and Q_2. For the function $f(x, y) = x + y$, evaluate $U(Q, f) - L(Q, f)$.

[1] ±1 for neatness and brevity.

(4) [4] Let R be a (closed) rectangle in R^2. Prove that if f and g are two integrable functions in R, so is $f+g$, and (!)

$$\int_R f+g = \int_R f + \int_R g$$

(5) [5] If $f : I \times I \to \mathbb{R}$ is C^1, show that the function $\mathcal{I}_f : I \to \mathbb{R}$, defined by

$$\mathcal{I}_f(y) = \int_0^1 f(x,y)dx$$

is differentiable, and

$$\frac{d\mathcal{I}_f}{dy}(y) = \int_0^1 \frac{\partial f(x,y)}{\partial y} dx$$

(6) [3] Consider the map $\vec{\gamma} : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$\vec{\gamma}(r,\theta) = (r\cos\theta, r\sin\theta)$$

(a) Is $\vec{\gamma}$ a C^1 map?

(b) Find $\tilde{U} \subset \mathbb{R}^2$ and $U \subset \mathbb{R}^2$ such that $\vec{\gamma}|_{\tilde{U}}$ is a C^1 diffeomorphism onto U and U is "as big as possible" (You don't have to make this precise, but do your best.)..

(7) [4] Let $GL(2,\mathbb{R})$ $\left(\subset M(2,\mathbb{R})\right)$ denote the set of invertible 2×2 matrices. Show that this is not connected. Let Inv be the map $GL(2,\mathbb{R}) \to GL(2,\mathbb{R})$ given by taking the inverse:

$$Inv(A) = A^{-1}$$

Show that derivative at $-I_2$ is

$$DInv(-I_2)[a] = -a, \ a \in M(2,\mathbb{R})$$

Here I_2 is the identity matrix, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Recall that the derivative of Inv at any matrix A is a linear map $M(2,\mathbb{R}) \to M(2,\mathbb{R})$.

Part B: Maximum Marks 15/20 ± 1[2]

(8) Let R be the closed rectangle $R = \{(x,y) | 0 \le x \le 1,\ 0 \le y \le 2\}$. Let $S \subset R \subset \mathbb{R}^2$ be the closed parallelogram

$$S = \{(x,y) | 0 \le x \le 1,\ x \le y \le x+1\}$$

Let $f(x,y) = y - x$.

(a) [4] Evaluate $\int_S f$. You can use any result proved in the course *except* change of variables.

(b) [6] Evaluate $\int_S f$ using a suitable change of variables and Fubini to reduce it to doing one-dimensional integrals.

(9) [6] Let f be a continuously differentiable (i.e., C^1) real-valued function on the open disc $D \equiv \{(x,y) | x^2 + y^2 \le 1\} \subset \mathbb{R}^2$, and suppose that $f_x^2 + f_y^2 \le 1$ everywhere on D. For $\vec{v}_1, \vec{v}_2 \in D$ find a bound for $|f(\vec{v}_1) - f(\vec{v}_2)|$ in terms of $||\vec{v}_1 - \vec{v}_2||$, where for any vector $\vec{v} = (a,b)$, $||\vec{v}||^2 \equiv a^2 + b^2$. Justify all steps. (Here $f_x = \frac{\partial f}{\partial x}$ etc.)

(10) [1+3] For $u, v > 0$, let $V(u,v)$ be the area of the rectangle

$$S(u,v) = \{(x,y) | |x| \le u, 0 \le y \le v\}\ .$$

Compute the "mixed partial derivative"

$$\frac{\partial^2 V}{\partial u \partial v}(u,v)$$

Let f be a continuous function on $\mathbb{R}^2$, and let F be defined by

$$F(u,v) = \int_{S(u,v)} f = \int_{|x| \le u, 0 \le y \le v} f(x,y)\,dx\,dy$$

Compute

$$\frac{\partial^2 F}{\partial u \partial v}(u,v)$$

in terms of f.

[2] ± 1 for neatness and brevity.