CMI Calculus 2022 (MU2103) Endsem

09:30-12:30, Monday, 21 November 2022

Write clearly at the start of the answer book: *Calculus Endsem. Your name and roll number.* Number the pages. You have to justify your steps.

`Maximum Marks 45/50 (±1 for clarity, neatness and brevity).`

All functions, vector fields, forms, etc. are smooth unless otherwise specified.

(1) Let $f : [0, \infty) \to \mathbb{R}$ be a non-negative continuous function such that $f(r) > 0,\ r < 1$ *and* $f(r) = 0,\ r \geq 1$. Let $S_f \subset \mathbb{R}^3$ be the "solid of revolution"

$$\{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 \leq 1 \text{ and } 0 \leq z \leq f(\sqrt{x^2 + y^2})\}$$

(a) [1] Is S compact? Briefly explain why.

(b) [3] Outline an argument that S_f is Jordan-measurable.

(c) [6] Prove that the volume of S_f is the integral under the graph of f, i.e.,

$$\int_{\mathbb{R}^3} \chi_{S_f}(x, y, z) dx dy dz = \int_{\mathbb{R}^2} f(\sqrt{x^2 + y^2}) dx dy$$

Quote precisely any result that you are using, and explain how it applies here.

(d) [8] Use a suitable change of variables to prove that

$$\int_{\mathbb{R}^2} f(\sqrt{x^2 + y^2}) dx dy = 2\pi \int_0^1 f(r) r dr$$

State the change of variables result that you are using, and carefully show how exactly it is applied. **The explanation has to be brief, clear and to the point.**

(e) [2] Evaluate the integral when $f(r) = \sqrt{1 - r^2},\ r \leq 1$.

(f) [3] Use this to compute the volume of the ball of radius 1 in $\mathbb{R}^3$. Justify your steps.

(2) [4] Let f be a function on $\mathbb{R}^2$; let $\vec{v} = \textit{grad } f$. Prove that given $P \neq Q \in \mathbb{R}^2$,

$$\frac{|f(Q) - f(P)|}{||Q - P||} \leq \int_0^1 ||\vec{v}(P + t(Q - P))|| dt$$

Here given $(a,b) \in \mathbb{R}^2$, $||(a,b)|| = \sqrt{a^2+b^2}$. Exhibit a function f and $P \neq Q$ for which equality is attained.

(3) [6] Consider the following vector field on $\mathbb{R}^2$:

$$\vec{v} = (-y^3, x^3)$$

Consider the closed unit disc $D = \{(x,y) | x^2 + y^2 \leq 1\}$. Compute the integral

$$\int_D curl\ \vec{v}$$

(You will have to make a change of variables, but you don't have to justify the steps.)

Check that Stokes' Theorem holds by computing the "line integral of $\vec{v}$ counterclockwise around the boundary circle". Hint: first show that the line integral reduces to: $\int_0^{2\pi} (\sin^4\theta + \cos^4\theta) d\theta$.

7 (4) [2] Let V be a n-dimensional vector space and B a bilinear form representing an inner product. What is $Alt^2(B)$?

(5) Consider the following 1-forms on $\mathbb{R}^2$:

$$\eta_1 = xdx + ydy, \ \eta_2 = -ydx + xdy$$

(a) [3] Compute $d\eta_i$, $i = 1,2$. When possible, find a function f_i such that $df_i = \eta_i$.

(b) [3] Consider the map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$\phi(r,\theta) = (r\cos\theta, r\sin\theta)$$

Compute $\phi^*\eta_i$, $i = 1,2$.

(c) [2] Verify $\phi^*(\eta_1 \wedge \eta_2) = \phi^*\eta_1 \wedge \phi_i^*\eta_2$. (needs some explanation)

(6) [2] Let $GL_+(2,\mathbb{R})$ $\left(\subset M(2,\mathbb{R})\right)$ denote the set of invertible 2×2 matrices with positive determinant. Is $GL_+(2,\mathbb{R})$ an open convex subset? Justify your answer.

(7) [5] Let $GL(n,\mathbb{R})$ $\left(\subset M(n,\mathbb{R})\right)$ denote the set of invertible $n \times n$ matrices. Consider the map

$$GL(n,\mathbb{R}) \to \mathbb{R}$$

$$A \mapsto (det\ A)^{-1}$$

Prove that the derivative of this map at $A = I_n$ is

$$M(n,\mathbb{R}) \ni B \mapsto -trace(B) \in \mathbb{R}$$