# End Semester Examination

B.Sc. II year

Analysis III

Maximum marks : 35

Duration: 120 min

Answer all the questions. First three questions carry Nine marks and last question is for Eight marks.

Write down the precise statements of the results you used in your solution along with your solution.

Your solutions should be legible, logical and complete in order to gain full points.

1. Suppose $\phi$ is a continuous bounded real function in the strip defined by $0 \le x \le 1, -\infty < y < \infty$. Prove that the initial-value problem $y' = \phi(x, y)$, $y(0) = c$ has a solution where 'c' is a constant.

[Hint: Fix $n$. For $i = 0, 1, 2, \dots, n$, put $x_i = \frac{i}{n}$. Let $f_n$ be a continuous function on $[0,1]$ such that $f_n(0) = c$, $f_n'(t) = \phi(x_i, f_n(x_i))$ if $x_i < t < x_{i+1}$, and put $\Delta_n(t) = f_n'(t) - \phi(t, f_n(t))$, except at the points $x_i$, where $\Delta_n(t) = 0$. Then $f_n(x) = c + \int_0^x [\phi(t, f_n(t)) + \Delta_n(t)]\, dt$.
choose $M > 0$ so that $|\phi| \le M$. check :

(a) $|f_n'| \le M$, $|\Delta_n| \le 2M$, $\Delta_n$ is R-integrable and $|f_n| \le c + M = M_1$ say, on $[0,1]$, for all $n$.

(b) Some $\{f_{n_k}\}$ converges to some $f$, uniformly on $[0,1]$

(c) On the rectangle $0 \le x \le 1$, $|y| \le M_1$, $\phi(t, f_{n_k}(t)) \longrightarrow \phi(t, f(t))$ uniformly on $[0,1]$.

(d) $\Delta_n(t) \to 0$ uniformly on $[0,1]$. Hence the result.]

(P.T.O)

2 (a) Suppose $0 < \delta < \pi$, $f(x) = 1$ if $|x| \le \delta$, $f(x) = 0$ if $\delta < |x| \le \pi$, and $f(x+2\pi) = f(x)$ for all $x$. Compute the Fourier coefficients of $f$. ~~Consol~~ conclude that $\sum_{n=1}^{\infty} \frac{\sin n\delta}{n} = \frac{\pi - \delta}{2}$ $(0 < \delta < \pi)$. Justify your answer.

(b) On $\mathbb{N}$ define $d(m,n) = \frac{|m-n|}{mn}$. Show that $d$ is a metric and the sequence $\{n\}_{n \in \mathbb{N}}$ is Cauchy sequence in this metric. Does this sequence converge in $\mathbb{N}$? Justify your answer. (show metric) [5+4].

3. Prove that Cantor set is a perfect set.

4. State Weierstrauss approximation theorem. Use it to prove the following: Let $A$ be an algebra of real continuous functions on a compact set $K$. If $A$ separates points on $K$ and if $A$ vanishes at no point of $K$, then prove that given a real function $f$, continuous on $K$, a point $x \in K$, $\exists$ a function $g_x \in B$ such that $g_x(x) = f(x)$ and $g_x(t) > f(t) - \epsilon$, $t \in K$ for any given $\epsilon > 0$. Here $B$ denotes the uniform closure of $A$.