Midterm- Monday 26/9/2022

Note: x, y denote variables.

1. Stat and prove Gauss Lemma. (5 marks)

2. Find the gcd d of $a_1 = 11 + 7i$ and $a_2 = 8 + i$ in $\mathbb{Z}[i]$. Is $(d) = (a_1) + (a_2)$? Explain.(5 marks)

3. Let R be a UFD with quotient field Q and f a primitive polynomial of positive degree in $R[x]$. Show that f is irreducible in $R[x]$ if and only if f is irreducible in $\mathbb{Q}[x]$. (5 marks)

4. Let R be a PID show that every ideal I in R can be expressed as a finite intersection of maximal ideals which are uniquely determined up to order. (5 marks) 4 in $\mathbb{Z}$ can't be written.

5. (a) Let R be an integral domain.
   i. Show that every prime is irreducible. Is the result true if we drop the assumption that R is an integral domain? (5 marks)
   ii. Determine whether $x^2 + 3x + 2$ is irreducible in $\mathbb{Z}[[x]] := \{\sum_{n \geq 0} a_i x^i : a_i \in \mathbb{Z}\}$. (5 marks)

   (b) Let $R = \mathbb{C}[\overline{x}, \overline{y}, \overline{z}] = \mathbb{C}[x, y, z]/(x^3 - yz)$. Prove or disprove: Is $(\overline{y}^2 - \overline{x}\overline{z})$ a prime element in R. (5 marks)

   (c) Let $R = \mathbb{C}[\overline{x}, \overline{y}, \overline{z}] = \mathbb{C}[x, y, z]/(x^2 + y^2 + y)$. Prove or disprove: $\overline{y}^4 + \overline{x}^3$ is irreducible in R. (5 marks)

6. (a) Show that every PID is a UFD. (5 marks)

   (b) Let R be a PID, Q its quotient field and x a variable. Let $S = \{f(x) = a_0 + a_1 x + \cdots + a_n x^n : a_0 \in R, a_1, \cdots, a_n, \in Q\}$.
   i. Show that S a ring. (5 marks)
   ii. Is S an integral domain? (5 marks)
   iii. Let $I = \{f(x) \in S | f(0) = 0\}$. Determine whether I is a prime ideal? Is it a maximal ideal. Explain. (5 marks)
   iv. Is S a principal ideal domain? If yes, prove it. If not, give an example of an ideal in S which is not principal. (5 marks)

7. Let A, B be commutative rings with identity element. Let $\phi : A \to B$ be a ring homomorphism.

   (a) Let $\mathfrak{p}$ be an ideal in A. Is $\phi(\mathfrak{p})B$ a prime ideal in B. If yes prove. If no give an example. (5 marks)

   (b) Let $\mathfrak{q}$ be an ideal in B. Is $\phi^{-1}(\mathfrak{q})$ a prime ideal in A. If yes prove. If no give an example. (5 marks)

   (c) Let R be a PID.
   i. Describe all the prime ideals in $R[x]$. Which of them are maximal. Explain how you derive your answer.(5 marks)
   ii. Suppose I and J are prime ideals in R. Describe all the maximal ideals in $R[x]/(IJ)R[x]$. (5 marks)

8. Let $\phi : \mathbb{Z}[x] \to \mathbb{Z}/2\mathbb{Z}$ be the composition of $\phi = \psi \circ \pi$, where $\psi : \mathbb{Z}[x] \to \mathbb{Z}$ is the map $\psi(x) = -1$ and $\pi : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is the quotient map. Determine $ker(\phi)$. (5 marks)

9. Let A be an integral domain, $R = A[x, y]$ and $S = R[x, y, x/y]$. Let $I = (x^3, xy^5y^7)$ be an ideal in R and $\phi : R \to S$ be the inclusion map. Prove or disprove: $I^{ec} = I$ (5 marks)

10. Let R be an integral domain. Suppose R is not a field. Let $U(R)$ denote the set of units in R. Show that $R \setminus \{U(R) \cup \{0\}\} \neq \emptyset$. Suppose that there exists no $y \in R \setminus \{U(R) \cup \{0\}\}$, such that for all $x \in R$, $y | (x - r)$ for some $r \in \{U(R) \cup \{0\}\}$. Show that R is not a Euclidean domain. (5 marks)