End-Semester- Wednesday 23/11/2023, Total Marks = 90

1. Let R be a non-zero ring (not necessarily commutative) with $1 \neq 0$.

   (a) Show that maximal (left) ideals always exist. (5 marks)

   (b) Suppose that R is a PID and $char(R) = p$ where p is a prime number. Suppose $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be two maximal ideals in $R[x]$. Suppose $R[x]/\mathfrak{m}_1 \cong R[x]/\mathfrak{m}_2$. Prove or disprove: $\mathfrak{m}_1 \not\cong \mathfrak{m}_2$. (5 marks)

2. Let $R_1 = \mathbb{Z}[\sqrt{3}]/(1 + 2\sqrt{3})$ and $R_2 = \mathbb{Z}/11\mathbb{Z}$ Determine whether $R_1 \cong R_2$. You need to explain the steps how you get the answer. (10 marks)

3. Let $R = \mathbb{Z}[x]/(f)$, where $f = x^4 + 49x^3 - 27x^2 + 50x - 2024$ and let $I = (7)$. Find all prime ideals of R that contain I. (Explain) (10 marks)

4. Let R be an integral domain and let $Q(R)$ denote its quotent field.

   (a) Let R be a UFD. Let $a, b \in R$ with $b \neq 0$. Suppose $a/b \in Q(R)$ satisifes a monic polynomial over R. Show that $a/b \in R$. (10 marks)

   (b) Let $R = \mathbb{C}[t^3, t^4, t^5]$ and $S = \mathbb{C}[t]$. Is every element of S algebraic over R? Is R a UFD? (5 marks)

5. Let K be a field of characteristic p. Consider the polynomial $x^{2p} - yx^p + y \in K(y)[x]$.

   (a) Show that f is irreducible over $K(y)$. (5 marks)

   ? (b) Let ~~K~~ L denote the splitting field of f (i.e. $f(x) = (x - a_1) \cdots (x - a_{2p})$ in ~~K~~ L). Determine $[L : K(y)]$. (5 marks)

6. Let $F \subseteq K$ be fields of characteristic 0. Suppose $a \in K$ and $[F(a) : F]$ is odd. Show that $F(a) = F(a^2)$. (5 marks)

7. Let R be a UFD of characteristic 0 and let Q its quotient field. Let be $f(x)$ be an irreducible polynomial in $Q[x]$. Determine the gcd of $f(x)$ and $f(x+1)$. Explain how you arrive at your answer. (5 marks)

8. Let $F = \mathbb{F}_2$. Explicitly describe all the irreducible factors of $f(x) = x^{16} - x$ over F. Explain how do you arrive at the answer. (10 marks)

9. Let p be a prime number and let $q = p^r$ for some integer $r > 0$. Let K be a field of order q.

   (a) Show that the non-zero elements of K for a cyclic group of order $q - 1$. (10 marks)

   (b) Show that the non-zero elements in $\mathbb{F}_3[x]/(x^2 + 1)$ form a cyclic group. What is the generator of the cyclic group? (5 marks)

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