CHENNAI MATHEMATICAL INSTITUE

Probability Theory
Date: 16th March 2022. Duration **14:00 to 16:30 hours**

Answer ALL questions. Give brief answers. Maximum credit for each question is 10 marks.

Q1. An urn contains 50 white balls and 50 black balls.
(a) Twenty balls are drawn at random from the urn (without replacement). What is the probability that there are 10 of each colour?
(b) One ball is drawn at a time, its colour is noted and replaced. This is repeated 20 times. What is the probability that among the balls drawn exactly 10 are white?
(c) All the balls are distributed in to 80 cells, where at most one ball of the *same* colour is allowed to occupy any cell. Find the probability that *all* the cells are occupied.

Q2. Urn A contains 12 red balls and 10 green balls. Urn B contains 10 red balls, 12 green balls and 7 blue balls. In an experiment, a ball is drawn at random from Urn A and is transferred to Urn B and then a ball is drawn from it.
(a) What is the probability that the second ball drawn is green?
(b) What is the probability that the first one drawn is green, given that the second ball drawn is green?
(c) Now the second ball is put into Urn A. What is the probability that the composition of the urns remain the same after the transfers?

Q3. Let X be a random variable with probability distribution function $F(x)$ defined as

$$F(x) = \begin{cases} 0, & x < 0 \\ 1/2, & 1/2 \leq x \leq 1 \\ 1/2 + C(e^{-1} - e^{-x}), & x \geq 1, \end{cases}$$

for a suitable constant C.
(a) Find the value of C.
(b) Show that X is neither discrete nor continuous.
(c) Find the distribution function of Y where $Y = \sqrt{X}$.
(d) Find the value of $P(1/3 \leq X \leq 3)$.

Q4. Suppose that X, Y are independent identically distributed discrete random variables which take values in $\{-1, 0, 1\}$ each value with equal probability. Let $Z = XY$ and $W = X + Y$. Find the mass function of Z. Are Z and W independent random variables?

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Q5. (a) Suppose that X is a continuous random variable with density function $f(x)$ which is nowhere zero. (a) Show that for any $a < b$, $P(a < X < b) > 0$.
(b) Show that given $\epsilon > 0$, there exists an $\alpha > 0$ such that $P(|X| > \alpha) < \epsilon$.
(c) Let X_n be the random variable defined as $X_n = \lfloor nX \rfloor / n$. Express the mass function of X_n in terms of $F(x)$.

Q6. (a) Suppose that X_1, X_2, X_3 are independent random variables which take values in positive integers. Suppose that $P(X_i = n) = p_i q_i^{n-1}$, $\forall n \geq 1$ where $0 < p_i < 1, q_i = 1 - p_i$, for $i = 1, 2, 3$. Show that

$$P(X_1 < X_2 < X_3) = \frac{p_1 p_2 q_2 q_3^2}{(1 - q_1 q_2 q_3)(1 - q_2 q_3)}.$$

(b) Suppose that X, Y are independent random variables with the density functions $f(x)$ and $g(x)$ defined as follows:

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq 1, \\ 0, & otherwise, \end{cases}$$

$$g(x) = \begin{cases} 2, & 0 \leq x \leq 1/2 \\ 0, & otherwise. \end{cases}$$

Find the distribution function of $Z = \min\{X, Y\}$.