CHENNAI MATHEMATICAL INSTITUE

Probability Theory

Final Examnation Duration: 3hrs

Answer ALL questions in Part A and any three queestions in Part B. Give brief answers.

PART A

1. There are 5 red balls and 6 black balls. The balls are indistinguishable except for their colours. They have to be distributed among 8 boxes, subject to the constraint that no two balls of the *same* colour are allowed to occupy the same box. The boxes are numbered 1 to 8. Find the probability that boxes $1, 2, 3$ receive two balls and the remaining boxes only one ball each.

2. In an experiment, n fair dice are rolled, where $n \geq 3$. Let $A_{ij}, 1 \leq < j \leq n$, be the event that the ith and the jth roll yield the same number. Show that the collection of events $\mathcal{A} := \{A_{ij} \mid 1 \leq i < j \leq n\}$ is not independent, but that any two members of $\mathcal{A}$ are independent.

3. In a game, N fair dice are rolled. The sum S of the numbers that show up on the dice is the score of the game. Suppose that N is a random number taking values in positive integers where $P(N = k) = (2/3)(1/3)^{k-1}$ for $k \geq 1$. (a) Find $P(S = 4)$.
(b) Find $P(N = 2|S = 4)$.

4. (a) Show that $\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$.
(b) Find the value of c so that $f(x) = ce^x(1 + e^x)^{-4}, x \in \mathbb{R}$, is the density function of a random variable. Also find the corresponding distribution function.

6. Let X, Y be independent random variables each being uniformly distributed on $[0, 1]$. Let $U = \min\{X, Y\}$. Find the density function of U and also find $E(U)$.

7. (a) Give an example of a discrete random variable which has no mean.
(b) Suppose that $0 < \lambda < 1$. Show that the function is $\lambda f + (1 - \lambda)g$ is the denisty function of a random variable if f, g are density functions of some random variables.
(c) Find examples of density functions f, g so that the product $h(x) = f(x).g(x)$ $\forall x \in \mathbb{R}$ is *not* a density function.

8. (a) Find the moment generating function $M_X(t) = E(e^{tX})$ of the Poisson random variable X with parameter λ.
(b)Let $X_1, \ldots, X_n$ be independent random variabales where X_j is Poisson with parameter λ_j, $1 \leq j \leq n$. Find the mean and variance of $Y = X_1 + \ldots + X_n$.

9. Let

$$f_{X,Y}(x,y) = \begin{cases} 1/x, & 0 < y \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find $f_X(x)$, $f_Y(y)$, and $f_{Y|X}(y|x)$.

10. Show that $\Gamma(1/2, 1/2)$-distribution is the same as the distribution of X^2 where X has the standard normal distribution $N(0,1)$.

PART B

11. (a) Determine the characteristic function $\phi_X(t)$ of the binomial random variable $X = Bin(n,p)$. Find a random variable Y such that $\phi_Y(t) = |\phi_X(t)|^2$.
(b) Let n be a positive integer. Show that $\phi(t) = e^{-t^2/2}(1+\cos t)^n/2^n$ is the characteristic function of a random variable Z.

12. Suppose that X, Y are independent random variables with $\Gamma(\lambda, s), \Gamma(\lambda, t)$-distributions respectively. Use characteristic functions to show that $X+Y$ is a $\Gamma(\lambda, s+t)$-random variable.

13. Let X and Y be independent exponential random variables with parameter 1. Find the joint density function of $U = X+Y$, $V = X/(X+Y)$ and deduce that V is uniformly distributed on $[0,1]$.

14. Find the value of c so that $f_{X,Y}(x,y) = c\exp(-x^2 + xy - 8y^2), x, y \in \mathbb{R}$, is the joint density function of the random variables X, Y. Also compute the density function $f_{Y|X}(y|x)$ of $Y|X = x$. Determine the random variable $E(Y|X)$.

$=1+e^n$ | $z = 1+e^t$