Discrete Mathematics Mid-semester Exam

March 2022

- You have 3 hours.
- Test contains two sections - 20 Multiple Choice Questions (each 4 marks) and 6 Subjective Questions (each 7 marks). Answer as much as you can. The maximum you can score in 100.
- Solutions can be handwritten or PDFs generated using LaTex. Solutions must be scanned/uploaded to the the Midsem Submission Link posted on moodle.
- Solutions should be uploaded by March 17, 14:00 PM IST after which the submission link will be automatically closed. This is a **hard deadline**.
- Collaboration of any kind if found will be treated as a serious case of academic dishonesty.

SECTION A Multiple Choice Questions

1. Which of the following statements is true?

   1. C_6 is not bipartite and K_3 is bipartite.
   2. C_6 is bipartite and K_3 is not bipartite.
   3. Both C_6 and K_3 are bipartite.
   4. Both C_6 and K_3 are not bipartite.

2. Let G be a graph with v vertices and e edges, Let M be the maximum degree of the vertices of G and m be the minimum degree of the vertices of G. Then, which of the following is true

   A. $\frac{2e}{v} \geq m$ and $\frac{2e}{v} \leq M$
   B. $\frac{2e}{v} < m$ and $\frac{2e}{v} \leq M$
   C. $\frac{2e}{v} < m$ and $\frac{2e}{v} > M$
   D. None of the above.

3. Consider the following statements.

   1. There exists a simple graph on 15 vertices each having degree 5.
   2. There is a simple graph with at least 2 vertices such that all vertices have distinct degrees.
   3. A vertex-induced subgraph induced by a non-empty subset of the vertex set of a complete graph is a complete graph.

   Which of the following is the correct?

   A. All the above statements are false.
   B. All the above statements are true.
   C. Statements 1,2 are false and 3 is true.
   D. 1 is false and 2,3 are true.

4. A simple graph is *nice* if if every vertex has the same degree. Consider the following statements:

   1. The undirected graph $G = (V, E)$ with $V = \{v_1, v_2, v_3\}$ and $E = \{(v_1, v_2), (v_2, v_3)\}$ is nice.
   2. K_n is nice for all $n \geq 1$ and C_n is nice for all $n \geq 3$.
   3. W_5, W_6 are nice and Q_n is nice for all $n \geq 1$.
   4. W_4 is nice and Q_n is nice for all $n \geq 1$.

Which of the above statements are true?

A. All 4 statements are true.
B. Only 2,4 are true
C. 1,2,4 are true.
D. Only 2 is true.

5. Let G be a simple graph with n vertices and e edges. Consider the graph B_G that has the same vertices as G and two vertices are adjacent in B_G if and only if they are not adjacent in G. Let us call B_G the *bad graph* corresponding to G. Consider the following statements.

   1. The bad graph corresponding to K_n has no edges.
   2. The union of the edge set of G and edges set of B_G has size $\frac{n(n-1)}{2}$.
   3. The number of edges in B_G is $\frac{n(n-1)}{2} - e$.
   4. The bad graph corresponding to C_5 is another cycle of length 5.

   Which of the above statements are true?

   A. All 4 statements are true.
   B. Only 1,2,4 are true
   C. 1,2,3 are true.
   D. Only 4 is true.

6. Which of the following statements about simple graphs are true:

   1. If G has more than $\frac{(n-1)(n-2)}{2}$ edges then G is connected.
   2. There are connected graphs with n vertices and $n-2$ edges.
   3. If G has e edges then G has at least $n-e$ connected components.
   4. If G has at least n edges then G has a cycle.

   Which of the above statements are true?

   A. All 4 statements are true.
   B. Only 1,3,4 are true
   C. 1,2,3 are true.
   D. Only 1 is true.

7. Which of the following statements is true?

   1. A bipartite graph with odd number of vertices does not have Hamiltonian circuit.
   2. The complete bipartite graph $K_{4,2}$ is Hamiltonian.
   3. The bad graph corresponding to K_4 is Hamiltonian.
   4. A 3×3 chessboard has a knight's tour.

8. Let $G = (V, E)$ be an undirected graph. Let $N_G(v)$ be the neighbourhood of vertex v in G.

   1. For every vertex v, $|N_G(v)| \leq \deg(v)$.
   2. For any $A, B \subseteq V$, $N_G(A \cup B) = N_G(A) \cup N_G(B)$.
   3. For any $A, B \subseteq V$, $N_G(A \cap B) \subseteq N_G(A) \cap N_G(B)$.
   4. There is a graph G and sets $A, B \subseteq V$ such that $N_G(A \cap B) \neq N_G(A) \cap N_G(B)$.

   Which of the above statements is true?

   A. All 4 statements are true.
   B. Only 1,3,4 are true
   C. 1,2,3 are true.
   D. Only 1 is true.

9. Consider the following statements:

   1. $|(0, 1)| = |(0, 1]|$.
   2. $|(0, 1)| = |[0, 1]|$.
   3. The cardinality of set S is smaller than the cardinality of the power set of S.
   4. The set of real numbers is countable.

   Which of the above statements is true?

   A. All 4 statements are true.
   B. Only 1,3,4 are true
   C. 1,2,3 are true.
   D. Only 1 is true.

10. Consider the following statements:

    1. The union of countable number of countable sets is countable.
    2. The Cartesian product of countable sets is countable.
    3. The set of all integers is countable.
    4. The set of positive rational numbers is countable.

    Which of the above statements is true?

    A. All 4 statements are true.
    B. Only 1,3,4 are true
    C. 1,2,3 are true.
    D. Only 1 is true.

11. Let $a_0 = 1$, and $a_{n+1} = 2\sum_{i=0}^{n} a_i$. Then what is the expression for a_n ?

    A. 2^n.
    B. $3 \cdot 2^n$.

C. $2 \cdot 3^{n-1}$.
D. None of the above.

12. If n is a positive integer then $8^n - 14n + 27$ is divisible by

A. 3.
B. 7.
C. both.
D. None of the above.

13. How many ways are there to arrange the digits $\{1, 2, 2, 3, 4, 5, 6\}$ so that identical digits are not in consecutive position?

A. 1230.
B. 1750.
C. 1800.
D. None of the above.

14. How many ways are there to select a subset $S \subseteq [15]$ such that S does not have two distinct elements a and b for which $a + b$ is divisible by three?

A. 252.
B. 378.
C. 126.
D. None of the above.

15. Let $n = rk$ where $r > 1$ is an integer. What is the largest coefficient in $(x_1 + x_2 + \ldots + x_k)^n$?

A. r^k.
B. $(r + k)!$.
C. $\frac{(rk)!}{r^k}$.
D. $\frac{(rk)!}{(r!)^k}$.
E. None of the above.

16. For all positive integer n, the expression $\binom{\sum_{k=1;k \text{ odd}}^{n} n}{k5^k}$ is same as

A. $\frac{5^n+4^n}{2}$.
B. $\frac{6^n-4^n}{2}$.
C. $\frac{6^n-(-4)^n}{2}$.
D. None of the above.

17. $S(n, 3)$ is equal to

A. $3^n - 2^n$.
B. $\frac{3^{n-1}+1}{2} - 2^{n-1}$.

C. $3 \cdot 2^n$.
D. None of the above.

18. The number of positive integers $k \leq 210$ which are relatively prime to 210 is

A. 50.
B. 48.
C. 65.
D. None of the above.

19. A function $f : [n] \to [n]$ is called acyclic if there are no cycle longer than one under the action of $[n]$. The numbe rof such acyclic functions

A. n^n.
B. $(n+1)^{n-1}$.
C. n^{n-1}.
D. None of the above.

20. Among any 502 positive integers, there are always two integers so that either their sum or their difference is divisible by 1000.

A. TRUE.
B. FALSE.
C. Insufficient information.

21. (5 points) Let k be an integer such that every man in a city is willing to marry exactly k of the women in the city and every woman in the city is willing to marry exactly k of the men. Also, suppose that a man is willing to marry a woman if and only if she is willing to marry him. Prove that it is possible to math men and women in the city so that everyone is matched with someone they are willing to marry.

22. (5 points) A *magic square* is an $n \times n$ array filled with n different symbols, each of which occurs exactly once in every row, and exactly once in every column. A *magic rectangle* is an $m \times n$ array for $m \leq n$ filled with n different symbols, in which each symbol occurs at most once in every row, and at most once in every column. Suppose we have a $r \times n$ magic rectangle with n symbols. Show that this can be extended to an $n \times n$ magic square.

23. (5 points) A king has invited his $2n$ ministers for a round table council. Every two ministers are either friends or enemies and each minister has no more than $n-1$ enemies among the other $2n-1$ ministers. We want to figure out whether the king can seat his ministers around the round table so that each minister has two friends as his neighbours.

    1. Show that this problem can be solved by determining whether there is Hamiltonian cycle in a graph G in which ministers are represented by vertices. Find out the edge relationship.
    2. Show that G that you constructed above has a Hamiltonian cycle.

24. (5 points) We choose $n+2$ numbers from the set $1, 2, \ldots, 3n$. Prove that there are always two numbers among the chosen numbers whose difference is more than n but less than $2n$.

25. (5 points) A child wants to walk up a stairway. At each step he moves up either one or two steps. Let $f(n)$ be the number of ways he can reach the n^{th} stair. Find an explicit formula for $f(n)$.

26. (5 points) Justify the following statement (i.e., explain whether this is true or false): The number of partitions of n into at most k parts is equal to the partitions of $n+k$ into exactly k parts.