HIAN

Discrete Mathematics End-semester Exam

May 2022

- You have 3 hours.
- Test contains two sections - 15 Multiple Choice Questions (each 3 marks) and 8 Subjective Questions (Total 56 marks). Answer as much as you can. The maximum you can score is 100. Of course the final grading is relative.
- Collaboration of any kind if found will be treated as a serious case of academic dishonesty.

SECTION A Multiple Choice Questions

1. Consider the following statements:

   (a) The set of real numbers between 0 and 1/2 is uncountable.

   (b) Real numbers not containing 0 in decimal representation is uncountable.

   (c) All bit strings not containing bit 0 is countable.

   (d) Positive rational numbers that cannot be written with denominator less than 4 is countable.

   Which of the above statements are true?

   (i) Only (a), (b) and (c) are true.

   (ii) Only (b), (c) and (d) are true.

   ✓(iii) All the above are true.

   (iv) None of the above are true.

2. Consider the following statements:

   (a) The set of real numbers that are solutions to $ax^2 + bx + c = 0, a, b, c \in \mathbb{Z}$ is countable.

   (b) There is a one-to-one correspondence between the set of positive integers and the power set of the set of positive integers.

   (c) The set of all finite bit strings are countable.

   (d) If sets A and B are countable then $A \cup B$ is also countable.

   Which of the above statements are true?

(i) Only (a), (b) and (c) are true.

(ii) Only (a), (c) and (d) are true.

(iii) All the above are true.

(iv) None of the above are true.

3. Let $m \geq 2$ and $a, b, c, d, m \in \mathbb{Z}$. Consider the following statements:

   (a) $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ implies that $a - c \equiv b - d \pmod{m}$.

   (b) $ac \equiv bc \pmod{m}$ implies that $a \equiv b \pmod{m}$.

   (c) $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ implies that $a^c \equiv b^d \pmod{m}$.

   (d) $a \equiv b \pmod{m}$ implies that $a^c \equiv b^c \pmod{m}$.

   Which of the above statements are true?

   (i) Only (a), (c) and (d) are true.

   (ii) None of the above are true.

   (iii) All the above are true.

   (iv) Only (a) and (d) are true.

4. Consider the following statements:

   (a) $\phi(10) = 5$ where $\phi(n)$ is the Euler's totient function that is the number of positive integers $\leq n$ that are co-prime to n.

   (b) For any integers a,b and $m \geq 2$, if $a \equiv b \pmod{m}$ then $\gcd(a, m) = \gcd(b, m)$.

   (c) If a and m are relatively prime then the multiplicative inverse of a modulo m exists.

   (d) The system $x \equiv 1 \pmod{5}, x \equiv 2 \pmod{6}, x \equiv 3 \pmod{7}$ of congruences has a solution.

   Which of the above statements are true?

   (i) Only (b), (c) and (d) are true.

   (ii) None of the above are true.

   (iii) All the above are true.

   (iv) Only (c) and (d) are true.

5. Which of the following statements is false?

   (a) A complete graph with more than two vertices is not Hamiltonian.

   (b) A directed graph has an Eulerian cycle if and only if every vertex has equal in-degree and out-degree, and all of its vertices with nonzero degree belong to a single strongly connected component.

   (c) Any Hamiltonian cycle can be converted to a Hamiltonian path by removing one of its edges.

   (d) An undirected graph has an Eulerian cycle if and only if every vertex has even degree, and all of its vertices with non-zero degree belong to a single connected component.

$6 - \frac{6\cdot 4}{2} + f = 2$

6. Consider the following statements about planar graphs:

   (a) A connected planar graph with 6 vertices each of degree 4 has eight plane faces(a.k.a regions) in a plane representation of graph G.

   (b) If G is a connected planar simple graph then G has a vertex of degree at most 5.

   (c) G be a planar graph with n vertices m edges and f faces then $f = m - n + 3$.

   (d) K_4 and Q_3 are planar.

   Which of the above statements are true?

   (i) None of the above are true.

   (ii) Only (a), (b) and (d) are true.

   (iii) All the above are true.

   (iv) Only (c) and (d) are true.

7. Let T be a tree with n vertices. Consider the following statements:

   (a) T has no cycles and $n-1$ edges.

   (b) There are edges in T that are not cut edges. Cut edges are those whose removal disconnects the tree.

   (c) There are vertices in T that are connected by more that one path.

   (d) T contains no cycle but addition of any new edge creates exactly one cycle.

   Which of the above statements are true?

   (i) None of the above are true.

   (ii) Only (a), (b) and (d) are true.

   (iii) Only (a) and (d) are true.

   (iv) All the above are true.

$\chi'(G) \in \{\Delta, \Delta+1\}$

8. Recall that $\chi(G)$ is the chromatic number of a graph G which is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices get the same color. Let us denote by $\chi'(G)$ the *edge-chromatic number* of a graph G which is smallest number of colors necessary to color each edge of G such that no two edges incident on the same vertex have the same color. Now, consider the following statements:

   (a) $\chi'(G) < \chi(G)$ for any graph G. [Hint: Consider $G = K_n$].

   (b) $\chi'(G) \geq \Delta(G)$ where $\Delta(G)$ is the maximum degree of any vertex in G.

   (c) The edge-chromatic number of a cycle on n vertices is 3 when n is odd.

   (d) When G is a graph with n vertices no more than $n/2$ edges can be colored with same color such that any two edges incident on the same vertex get different colors. [Hint: Use matchings]

   Which of the above statements are true?

(i) Only (b) and (d) are true.

(ii) Only (b), (c) and (d) are true.

(iii) Only (b) and (c) are true.

(iv) All the above are true.

9. A graph property retained whenever edges are added to a simple graph (without adding vertices) is called **monotone increasing** and a property that is retained whenever edges are removed from a simple graph (without removing vertices) is called **monotone decreasing**. Consider the following statements:

   (a) A graph being connected is a monotone increasing property.

   (b) A graph being non-connected is a monotone decreasing property.

   (c) A graph being planar is a monotone increasing property.

   (d) A graph being Hamiltonian(i.e., having a hamiltonian cycle) is a monotone increasing property.

   Which of the above statements are true?

   (i) Only (b) and (d) are true.

   (ii) Only (b), (c) and (d) are true.

   (iii) Only (a), (b) and (d) are true.

   (iv) All the above are true.

10. A traveling agent has to visit four cities, each of them five times. In how many different ways can he do this if he is not allowed to start and finish in the same city?

   (i) $\frac{20!}{(5!)^4} - 4 \cdot \frac{18!}{(5!)^3 \cdot 3!}$.   A,B,C,D.

   (ii) $\frac{20!}{(5!)^3 \cdot 4!} - 4 \cdot \frac{18!}{(5!)^4}$.

   (iii) $\frac{20!}{(5!)^3 \cdot 4!} - 4 \cdot \frac{20!}{(4!)^4}$.

   (iv) None of the above.

11. In how many different ways can we place eight identical rooks on a chess board so that no two of them attack each other?

   (i) 40000.   5! = 120   120

   (ii) 40850.   8!.   6! = 720   720

   (iii) 40320.   7! = 5040   5040

   (iv) None of the above.   8! = 40320   40320

12. How many ways are there to list the digits $\{1, 2, 2, 3, 4, 5, 6\}$ so that identical digits are not in consecutive positions?

   $\frac{7!}{2!} - 6!$ 2520   5040/2

   (i) 2000.

   (ii) 1800.   2520 − 720

   (iii) 2520.   1800

(iv) None of the above.

13. A simple closed form generating function of the sequence defined by $a_n = n^2 (n \geq 0)$ would be

$(1-x)^{-1} = 1+x+x^2 \cdots$

(i) $x/(1-x)^3$.

$x(1-x)^{-2} = x + 2x^2 + 3x^3 \cdots$

(ii) $x(x+1)/(1-x)^3$.

(iii) $(1+x)/(1-x)^3$.

$(1-x)^{-2} + x(+2(1-x)^{-3}) = 1 + 2^2x^1 + 3^2x^2 +$

(iv) None of the above.

14. Let $f(n)$ be the number of subsets of $[n]$ in which the distance of any two elements is at least three. A correct recurrence relation for $f(n)$ would be

(i) $f(n) = f(n-2) + f(n-3)$.

$\frac{1-x}{(1-x)^3} + \frac{2x}{(1-x)^{-3}}$

(ii) $f(n) = f(n-1) + f(n-2)$.

(iii) $f(n) = f(n-1) + f(n-3)$.

$= \frac{1+x}{(1-x)^{-3}}$

(iv) None of the above.

15. We select an element of $\{1, 2, \cdots, 100\}$ at random. Let A be the event that this integer is divisible by three and B be the event that this integer is divisible by 7. Are A, B independent?

$\mathbb{P}[A] = \frac{\lfloor \frac{100}{3} \rfloor}{100} = \frac{33}{100}$.

(i) Yes.

(ii) No.

$\mathbb{P}[B] = \frac{14}{100}$.

(iii) Insufficient information.

$\mathbb{P}[A \cap B] = \frac{5}{100}$.

SECTION B. Subjective Questions.

$x^2 \equiv 1 \pmod n$.

1. (3+5 points) Prove the following statements:

   (a) n is prime if and only if $\phi(n) = n - 1$.

   (b) n is prime if and only if $(n-1)! \equiv -1 \pmod n$.

2. (4+2.5+2.5 points) Use the number theory tools developed in class to show the following:

   (a) If $a \equiv b \pmod n$ then for all $c > 0$ such that $c|a$ and $c|b$, prove that $\frac{a}{c} \equiv \frac{b}{c} \pmod{\frac{n}{\gcd(n,c)}}$.

   Prove Euler's totient function :

   (b) Estimate $3^{45} \pmod{100}$.

   (c) Use Fermat's Little Theorem to find the last digit of 3^{100}.

$\varphi(100)$

$= 100$

3. (2+1+1+1+2+2+2 points) The degree sequence of a graph is the sequence of degrees of the vertices of the graph in non-~~decreasing~~ increasing order.

   (a) What is the degree sequence of the complete graph K_n?

(b) How many edges does a graph have if the degree sequence is 5, 2, 2, 2, 2, 1?

(c) Can there be a simple graph with the degree sequence 6, 5, 5, 4, 3, 3, 3, 2, 2?

(d) Can there be a simple graph with the degree sequence 7, 6, 4, 3, 3, 2?

(e) Can there be a simple graph with the degree sequence 3, 1, 1, 0? If yes, draw such a simple graph.

(f) The complementary graph of G(denoted by $\overline{G}$)has the same set of vertices as G and two vertices in $\overline{G}$ are adjacent if and only if they are non-adjacent in G. If the degree sequence of a graph G is $d_1, d_2, \ldots, d_n$ then what is the degree sequence of the complementary graph $\overline{G}$?

(g) Suppose there exists a simple graph with degree sequence $d_1, d_2, \ldots, d_n$ then show that there is a simple graph with degree sequence $d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$.

4. (a) (4 points) In a $2n \times 2n$ chessboard there are n rooks in each row and n rooks in each column of the board. Show that there exists $2n$ rooks that belong to pairwise distinct rows and columns. Hall's theorem; Perfect matching.

(b) (4 points) Prove that every tournament has a Hamiltonian path. Recall that a *tournament* is a directed graph obtained by assigning a direction for each edge in an undirected complete graph. Induction.

5. (4 points) Let A be an $n \times n(n \geq 2)$ matrix with 0, 1 entries, and at least $2n$ entries are equal to 1. Prove that A contains two entries equal to 1 so that one of them is strictly above and strictly on the right of the other. Generalise then

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6. (4 points) How many ways are there to select a subset $S \subseteq [15]$ such that S does not have two distinct elements a and b for which $a + b$ is divisible by 3?

7. (4 points) Let a_n be the number of permutations of length n in which the entry i is never immediately followed by the entry $i+1$. Prove that for $n \geq 3$, the recurrence relation $a_n = (n-1)a_{n-1} + (n-2)a_{n-2}$. PIE both?

8. (4 points) Prove that there is a tournament on n vertices that contains at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.