Analysis 2 Final

- Explain everything. You may use any standard result, but refer to it precisely and justify why it is applicable. Exception: for material from one variable calculus you may explain briefly. (You need not explain full details of proofs that you could have written last semester.)
- Each problem is worth 6-8 points, averaging roughly 7. Solve everything you can. There is some option built in because the exam will count out of at most 50 points. And whatever that total x will be (depending on exact weight of other components), I will count your best work out of problems worth x points.
- **Write your answers on loose sheets. Label each *side* you use (not just sheets) as follows.** At the top of each side of each page (i) write on the left your name and roll number (ii) write on the right: Problem i page x of y.
- **Do NOT crumple any sheet because all papers have to pass through a copier for scanning.**
- Optional Quiz 5 will ask you to correct your own final (with comments) and fix all mistakes.You may do the same for all quizzes as well. Details will be sent by email. Deadline: May 31.

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1. True or false with justification. Given: $p \in$ open U in some Euclidean space and a function $f : U \to \mathbb{R}^n$.

   (i) Suppose f is differentiable at p and γ is the "straight line" curve based at p, namely $\gamma(t) = p + tv$, where v is any vector. Then $f \circ \gamma$ is differentiable at 0 and $(f \circ \gamma)'(0) = f'(p)(v)$.

   (ii) Suppose f has all directional derivatives at p. Then f is continuous at p.

   (iii) Suppose f has all partial derivatives at p and they are continuous at p. Then f is continuous at p.

   (iv) Suppose f has all partial derivatives at p and they are continuous at p. Then the total derivative of f exists at p and is continuous at p.

   (v) Short answer: Let f be a continuous function from an open ball $B = B(a, r)$ in $\mathbb{R}^2$ to $\mathbb{R}$. Is $f(B)$ necessarily open? Is $f(B)$ necessarily bounded? If f is from all of $\mathbb{R}^2$ to $\mathbb{R}$, does the answer to either question (still about $f(B)$) change?

2. Let U be an open set in $\mathbb{R}^n$ and f an *infinitely differentiable* function from U to $\mathbb{R}$. Complete the following discussion by filling in the blanks with the most information that you can about analyzing f for local extrema. You need not repeat the text below. Just label your answers A through E. For D and E, you can simply refer to relevant standard fact(s). E.g., if you are using a theorem, then you can just state the name of the theorem and explain briefly.

   Suppose f has a local extremum at a point p. Then the following *must* be true about suitable derivative(s) of f: __A__. If the extremum at p is a local maximum, then the following *must* be true in addition to A about suitable derivative(s) of f: __B__. If the assumption of *infinite differentiability* can be relaxed, state the weakest assumption(s) that keeps the respective conclusion intact (separately for A and B if necessary) and explain briefly: __C__.

   Conversely, if the following condition(s) is(are) true about the function f __D__, then f *must* have a local maximum at the point p. If the assumption of *infinitely differentiability* can be relaxed, state the weakest assumption(s) you need to keep the conclusion intact and explain briefly where any such assumption is used. __E__.

3. Suppose $f : \mathbb{C} \to \mathbb{C}$ is *complex differentiable* at $z_0 = a + ib$ where $a, b \in \mathbb{R}$. Show that when considered as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$, f is differentiable at (a, b) and the partial derivatives of f at (a, b) satisfy the Cauchy Riemann conditions at z_0. (Notation: for $z = x+iy$, let $f(z) = f(x+iy) = u(x, y)+iz(x, y)$ where u and v are real valued functions of real variables x and y.)

4. The function $g : \mathbb{R}^2 \to \mathbb{R}$ is defined by $g(0,0) = 0$ and $g(x,y) = (x^2 + y^2)\sin\left(\frac{1}{x^2+y^2}\right)$ otherwise. Show from first principles that g is differentiable at $(0,0)$. Then do the same using chain rule. Is g a C^1 function?

5. For $f(x,y) = (g(x,y), h(x,y)) = (e^{x+2y}, (x+3)(y+4))$ find $Df(0,0)\left(\begin{bmatrix}6\\7\end{bmatrix}\right)$. Find a unit vector u with maximum $|D_u h(0,0)|$ (usual Euclidean norm). How will you find the operator norm $||Df(0,0)||$? Just sketch a suitable method that can be used to calculate the answer. Do not carry it out.

6. Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function with the following properties: g has precisely two zeros at a, b with $g'(a) > 0$, $g'(b) < 0$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = \int_x^y g(t)dt$. Find local max, local min and saddle points of f. You may make any additional assumption(s) if you need, which you must specify clearly with reason.

7. (i) Find the equation(s) describing the tangent space at (1,1,0) to level set M defined below. Find a basis for this tangent space.

$$M = \{(x,y,z)|f(x,y,z) = x^4 + 2y^2 + 2z^2 = 3\} \subset \mathbb{R}^3.$$

(ii) Near each point of M can one locally write some coordinate(s) in terms of others?

(iii) A subset M of a Euclidean space is called a C^1 manifold of dimension d if the following is true: for each $p \in M$, there is an open subset U of M (U is open *in* M) and a bijection $\pi : U \to$ some *open* set of $\mathbb{R}^d$ such that both π and π^{-1} are C^1 functions. Show that M is a C^1 manifold. (You have to specify d. What tool can you use to produce such C^1 functions?)

8. Let $S = \{(x,y)|h(x,y) = y^2 + x^4 - x^3 = 0\} \subset \mathbb{R}^2$

(i) Show that S is compact.

(ii) Does $f(x,y) = x$ have global extrema on S? If so, find them. What does the Lagrange multiplier method give?

(iii) Find all points on S near which it is not possible to solve for y in terms of x. (Do later for yourself: for each point p you found, is x a function of y near p? Is yes, is this function differentiable?)

Ideally I would like you to stop here. Try 9 and 10 now ONLY if you found a lot of 1–8 difficult.

9. (i) Give an example of a function from $\mathbb{R}^n \to \mathbb{R}^n$ that is locally invertible everywhere but not injective.
(ii) Carefully state the implicit function theorem and sketch how to deduce it from the inverse function theorem. You do not need to explain all points in detail, but do not *skip* any point.

10. Here are some things to think about on your own. **Do this after the exam.** (i) An interesting example of optimization with multiple constraints where the Lagrange multiplier method fails to detect an extremum. (ii) Chain rule for higher derivatives? (iii) In the situation of problem 1, suppose there is a linear map T such that for every differentiable curve γ based at p, $f \circ \gamma$ is differentiable at 0 with $(f \circ \gamma)'(0) = T(\gamma'(0))$. Then is it true that f is differentiable at p (which would force $f'(p) = T$)?