Algebra 2, Jan–May 2022, Midterm exam
2022-Mar-15, 09:30-12:00 IST.

Please see the `Instructions` file in moodle for instructions. If you have questions, please ask in zoom. Submission deadlines: scans to be sent to my gmail address by 12:10; upload to moodle by 12:45.
This exam is worth 150 marks.
Contribution towards your final grade calculation: your marks $\times$ 30/150.
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1. Please read all the questions; some questions may contain hints for other questions.
2. Please write your roll number on all the pages.
3. Please provide justification for all your answers including the True/False questions.
4. You may use results from the lectures and the textbook that were proved in class, or were included in the exercises, unless solving the exercise is the content of a question in this exam.

Notation

1. $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$: the set (or field, if applicable) of non-negative integers $\{0, 1, \dots, \}$, of integers, of real numbers, of complex numbers.
2. F a field, typically $\mathbb{R}$ or $\mathbb{C}$
3. V a finite-dimensional vector space over F.
4. $p_T(t), p_A(t)$: characteristic polynomial of a linear map T or a matrix A.
5. $m_T(t), m_A(t)$: minimal polynomial of a linear map T or a matrix A.

Questions

1. (15 marks) Let A be a real square matrix with $A^t = -A$. Prove or disprove the following statement: The form $\mathbf{x}^t A^2 \mathbf{y}$ is negative definite.
2. (5 marks) True/False: Let A be a diagonalizable 3×3 matrix. Let $f(X) = X^{17} + 3X$. Then $\deg m_A(t) = \deg m_{f(A)}(t)$.
3. A group G is said to be *cyclic* if there exists $g \in G$ such that $G = \{e, g, g^{-1}, g^2, g^{-2}, g^3, g^{-3}, \dots\}$.
    (a) (5 marks) Using an appropriate proposition from Artin, Chapter 2, show that every subgroup of G is cyclic.
    (b) (5 marks) Show that if $\phi : G \to G'$ is a surjective homomorphism, then G' is cyclic.
    (c) (5 marks) Show that the Klein four group (with the description given in Equation (2.4.4) in Artin Chapter 2) is Abelian but not cyclic.
4. (15 marks) Let A be a 2×2 real matrix with eigenvalues 2 and -1. Show that there is a proper subspace W of $\mathbb{R}^2$ such that for all vectors $\mathbf{x} \in \mathbb{R}^2 \setminus W$, the sequence $2^{-k}A^k\mathbf{x}, k \geq 1$ converges to an eigenvector of A. (Distance between two vectors v, w is the Euclidean length of $v - w$.)
5. Let A be a 3×3 real symmetric matrix with eigenvalues $1, 0, -1$. Determine which of the following statements are true.

(a) (5 marks) $\ker A \cap \mathrm{Im}(A) = \{0\}$.

(b) (5 marks) $\ker A \perp \mathrm{Im}(A)$.

(c) (5 marks) $v \perp Av$ for all $v \in \mathbb{R}^3$.

6. Let T, T' be linear operators on a finite-dimensional complex vector space V. Say that T and T' are *friendly* if there exists a basis of V in which both T and T' are given by diagonal matrices.

   (a) (15 marks) Assume that T and T' are diagonalizable. Show that they are friendly if and only if $TT' = T'T$.

   (b) (5 marks) Show that one of the directions of the above "if and only if" statement does not hold if we drop the hypothesis that T and T' are diagonalizable.

7. (15 marks) Show that the function

$$\phi : S_n \to \mathrm{GL}_n(\mathbb{Q})$$

with $\phi(\sigma) = (a_{i,j})$ where for each $1 \leq i \leq n$,

$$a_{i,j} = \begin{cases} 1, & \text{if } j = \sigma(i) \\ 0, & \text{otherwise.} \end{cases}$$

is an injective group homomorphism.

8. Let V be the real vector space of polynomials of degree at most 2. Let $\langle -, - \rangle$ be the symmetric bilinear form

$$\int_0^1 f(x)g(x)\mathrm{d}x$$

   (a) (10 marks) Determine the matrix of the form with respect to the basis $\{1, x, x^2\}$.

   (b) (10 marks) Do the Gram-Schmidt orthonormisation process to the basis $\{1, x, x^2\}$ and get an orthonormal basis of V.

9. (15 marks) Let A and B be 3×5 and 5×3 matrices respectively. Show that

$$\{\lambda \mid \lambda \neq 0, \lambda \text{ is an eigenvalue of } AB\} = \{\lambda \mid \lambda \neq 0, \lambda \text{ is an eigenvalue of } BA\}$$

10. (5 marks) Consider the bilinear form on $\mathbb{R}^2$ given by the matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

Find a subspace W such that $W \neq (W^\perp)^\perp$ with respect to this form.

11. (10 marks) Let T be a normal operator on a complex vector space V. Prove or disprove the following statement: T is Hermitian if and only if its eigenvalues are real.