Algebra 2, Jan–May 2022, Final exam (i, k, ℓ)

2022-May-21, 09:30-12:30.

1. This is a closed-book exam. Electronic devices are not allowed inside the room.
2. This exam is worth 160 marks. Contribution towards your final grade calculation: your marks × 40/160.
3. Please provide justification for all your answers including the True/False questions.
4. You may use results from the lectures and the textbook that were proved in class, or were included in the exercises, unless solving the exercise is the content of a question in this exam.

Questions

1. Let $n \geq 2$ be an integer and $\sigma, \tau \in S_n$.
   (a) (5 marks) Show that if σ and τ are conjugates of each other, then their cycle decompositions have the same order, i.e., for each k, the number of k-cycles in their cycle decompositions is the same for both σ and τ.
   (b) (10 marks) Prove the converse.
   (c) (5 marks) Determine the class equation of S_4.
   (d) (10 marks) Show that if $n \geq 4$, A_n is generated by 3-cycles.
   (e) (5 marks) Determine whether the 3-cycles form a single conjugacy class in A_4.

2. (5 marks) Let $A \in \mathrm{GL}_4(\mathbb{C})$ be an element of finite order, with $A \neq I_4$. Prove or disprove the following statement: 1 is the only eigenvalue of A.

3. (10 marks) Construct an orthonormal basis of $\mathbb{R}^4$ starting with

$$v_1 := \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

4. Below we give another proof of Sylow's first theorem. Let G be a finite group and p a prime number that divides $|G|$. Assume by induction that the theorem holds for G' for all groups G' with $|G'| < |G|$.
   (a) (5 marks) State Sylow's first theorem.
   (b) (5 marks) Show that if G has a subgroup H such that $[G : H]$ is not divisible by p, the theorem holds for G. H proper.
   (c) (15 marks) Hence assume that for every subgroup H of G, $[G : H]$ is divisible by p. Let Z be the centre of G. Each of the parts below is worth 5 marks.
      1. Show that p divides $|Z|$.
      2. Show that G has a normal subgroup H of order p.
      3. Show that the theorem holds for G. (Hint: Consider G/H.)

5. Let G be a group and F a field. A *(finite-dimensional) representation* of G over F is a (finite-dimensional) F-vector-space V together with a group homomorphism $G \longrightarrow GL(V)$. (I.e., there is an action of G on V, which is as invertible linear transformations, not merely as bijective functions.) Let V be a representation of G. By V^G, we mean the set

$$\{v \in V \mid gv = v \text{ for all } g \in G\}.$$

(a) (5 marks) Show that V^G is a subspace of V.

(b) (5 marks) Let $G := \mathbb{Z}/2\mathbb{Z}$ act on $V := \mathbb{C}^2$ with $\bar{1}v = -v$. Determine V^G.

(c) (10 marks) Let $G := \mathbb{Z}/2\mathbb{Z}$ act on $V := \mathbb{C}^2$ with $\bar{1}e_1 = e_2$ and $\bar{1}e_2 = e_1$, where e_1, e_2 is the standard basis for $\mathbb{C}^2$. Determine V^G.

(d) (15 marks) Show that V/V^G is a representation of G.

6. (20 marks) Let H be a subgroup of S_n. Show that the following are equivalent:

   1. $\{(1,2), (1,2,\cdots,n)\} \subseteq H$.
   2. $(k, k+1) \in H$ for each $1 \leq k \leq n-1$.
   3. Every transposition of S_n is in H.
   4. $H = S_n$.

7. (10 marks) Let V be a five-dimensional complex vector space, and let T be a linear operator on V with characteristic polynomial $(t-\lambda)^5$. Suppose that the rank of the operator $T-\lambda I_5$ is 2. Determine the possible Jordan forms for T.

8. (10 marks) Determine all the groups of order 6 up to isomorphism. $\mathbb{Z}/6\mathbb{Z}$, $S_3 \cong D_6$.

9. (10 marks) The characteristic polynomial of the complex matrix

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & ? & ? \end{bmatrix}$$

is divisible by t. Determine the missing entries (not necessarily uniquely determined). (You must show how you arrive at the answer, not just give the answer.)