Analysis -I

Mid-Sem examination on Analysis.

3 Nov 2021: 10:30am

Answer all questions.
Each part in question 1 has 6 marks (total 36 marks for question 1).
Each part in question 3 has 6 marks (total 30 marks for question 3).
Questions 3,4,5 have 6 marks each
Question 6, parts (i) and (iii) have 3 marks each and part (ii) has 10 marks.

If you use a result proved in class or notes, refer to it precisely stating the same.

All answers must be handwritten and the answer has to be uploaded to moodle. If you have difficulty with moodle, send the answer sheet to aniruddh.math@gmail.com

The test is for 2 hours and half an hour grace period and half an hour for taking a photocopy and uploading to moodle. Thus you must upload before 1.30pm I will be on zoom at the usual link for our class till 11.30am. Try to ask questions before that. Later, you can send me question on WhatsApp shared group, where I have been added yesterday.

(1) Let $\{a_n : n \geq 1\}$ be a sequence of real numbers such that $0 \leq a_n \leq 1$ for all $n \geq 1$. Let $F_m = \{a_n : n \geq m\}$, $u_m =$ l.u.b. F_m and $v_m =$ g.l.b. F_m. Show that

(i) $F_{m+1} \subseteq F_m$ for all $m \geq 1$.

(ii) $0 \leq v_m \leq v_{m+1} \leq u_{m+1} \leq u_m$ for all $m \geq 1$.

(iii) $\alpha = \lim_{m\to\infty} u_m$ and $\beta = \lim_{m\to\infty} v_m$ exist.

(iv) $\forall \epsilon > 0\ \exists m_1, m_2$ such that $a_{n_1} \leq \alpha + \epsilon$ for all $n_1 \geq m_1$ and $a_{n_2} \geq \beta - \epsilon$ for all $n_2 \geq m_2$.

(v) $\forall m \geq 1, \forall \epsilon > 0, \exists n_1, n_2,\ \ n_1 \geq m,\ n_2 \geq m$ such that $a_{n_1} \geq \alpha - \epsilon,\ a_{n_2} \leq \beta + \epsilon$.

(vi) If $\alpha = \beta$, then $\alpha = \lim_{n\to\infty} a_n$.

(2) **Let $f : (0, 1] \mapsto \mathbb{R}$ be a continuous function. Suppose for any sequence of real numbers $\{a_n : n \geq 1\}$, $0 < a_n < 1$ such that a_n converges to 0, $\lim_{n\to\infty} f(a_n)$ exists. Recall our convention that limit, when it exists, is a real number. Show that**

(i) **If $\{u_n : n \geq 1\}$, $\{v_n : n \geq 1\}$ are two sequences converging to 0, then the sequence $\{w_n : n \geq 1\}$ defined by $w_{2n} = v_n$ and $w_{2n-1} = u_n$, also converges to 0.**

(ii) **If $\{u_n : n \geq 1\} \subseteq (0, 1)$, $\{v_n : n \geq 1\} \subseteq (0, 1)$ are two sequences converging to 0, then $\lim_{n\to\infty} f(u_n) = \lim_{n\to\infty} f(v_n)$.**

(iii) **$\lim_{x\downarrow 0} f(x)$ exists.**

(iv) **$g : [0, 1] \mapsto \mathbb{R}$ defined by**

$$g(x) = \begin{cases} f(x) & \text{if } x \in (0, 1] \\ \lim_{t\downarrow 0} f(t) & \text{if } x = 0 \end{cases}$$

is a continuous function on $\mathbb{R}$.

(v) **g and f are uniformly continuous functions on $[0, 1]$ and $(0, 1]$ respectively.**

(3) **Let $\{d_n : n \geq 1\}$ be a sequence of real numbers satisfying the Cauchy criterion: $\forall \epsilon > 0$, $\exists n_0$ such that**

$$|d_m - d_n| \leq \epsilon \;\; \forall m, n \geq n_0.$$

Show that $\lim_{m\to\infty} d_m$ exists.

(4) **Let $E = [a, b]$ with $a, b \in \mathbb{R}$, $a < b$. Let $f : [a, b] \mapsto \mathbb{R}$ be a continuous function. Suppose $\min\{f(a), f(b)\} < \max\{f(a), f(b)\}$ and let c be such that**

$$\min\{f(a), f(b)\} < c < \max\{f(a), f(b)\}.$$

Show that $\exists z \in [a, b]$ such that $f(z) = c$.

(5) **Let $E = [a, b]$ with $a, b \in \mathbb{R}$, $a < b$. Let $f : [a, b] \mapsto \mathbb{R}$ be a continuous function. Let $F = \{f(x) : x \in E\}$. Show that F is also a closed bounded interval.**

(6) Let $\{u_n : n \geq 0\}$ be a sequence of real numbers. Suppose $T > 0$ is such that

$$\left(\limsup_{n\to\infty} \sqrt[n]{|u_n|}\right) < \frac{1}{T}$$

Show that

(i) $\sum_{n=0}^{\infty} |u_n||x|^n < \infty$ for all $x \in [-T, T]$

(ii) $f : [-T, T] \mapsto \mathbb{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty} u_n x^n$$

is a continuous function.

(iii) f (defined above) is uniformly continuous on $[-T, T]$.

END
