Analysis -I
End-Sem examination on Analysis.
31 Dec 2021: 9:30am-12:30am

Answer all questions.

All answers must be handwritten and the answer sheet has to be uploaded to moodle. If you have difficulty with moodle, send the answer sheet by email to `rlkcmi_endsem@gmail.com` **with subject: "BScI-End_Sem_Exam".**

You must write your name and your phone number on the answer sheet.

The test is for 2 and a half hours. You also have half an hour grace period and half an hour for taking a photocopy and uploading to moodle. Thus you must upload before 1.00pm.

Marks for questions are marked in Green in each question.

I will be on zoom at the usual link for our class from 9.15am to 11.00am. Try to ask questions before 11.00am. Later, you can send me question(s) on WhatsApp shared group.

In your answers in Part 1 (Questions 1,2), you can only use definitions of

(i) convergence of sequences, series , (ii) continuity, (iii) differentiability and (iv) Riemann Integration of functions.

If you wish to use any other result done in the class, you should state and prove the same.

You may use conclusion(s) of other questions or parts thereof in this question paper, even if you have not answered them (do quote question number in the explanation).

You may use the notation used in class for Riemann integration (given here for easy recall): Let $f : [a, b] \mapsto \mathbb{R}$ be a function. A partition P of $[a, b]$ is a finite subset of $[a, b]$ such that $a, b \in P$. We will index elements of P as

$$P = \{x_0, x_1, \ldots, x_{m-1}, x_m : \ a = x_0 < x_1 < \ldots < x_{m-1} < x_m = b\}$$

for some integer m, $1 \leq m < \infty$. Let

$$U(a, b, f, P) = \sum_{k=1}^{m} (x_k - x_{k-1}) \sup\{f(y) : \ x_{k-1} \leq y \leq x_k\}$$

$$L(a, b, f, P) = \sum_{k=1}^{m} (x_k - x_{k-1}) \inf\{f(y) : \ x_{k-1} \leq y \leq x_k\}$$

$$L(a, b, f) = \sup\{L(a, b, f, P) : P \text{ a partition of } [a, b]\}$$

$$U(a, b, f) = \inf\{U(a, b, f, Q) : Q \text{ a partition of } [a, b]\}.$$

Recall that f is said to be Riemann integrable on $[a, b]$ if $L(a, b, f) = U(a, b, f)$

Part 1:

(1) Show that f is Riemann integrable on $[a, b]$ if and only if for any $\epsilon > 0$, $\exists$ partition P of $[a, b]$ such that [15]

$$(U(a, b, f, P) - L(a, b, f, P)) < \epsilon.$$

(2) Let $\{a_n : n \geq 0\}$ be such that

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1.$$

Let $\theta \in (0, 1)$ be fixed. Show that [7+7+7+7+7]

(i) For each $x \in [-\theta, \theta]$, the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent.

(ii) The function f defined by $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $x \in [-\theta, \theta]$ is a continuous function on $[-\theta, \theta]$.

(iii) f defined above is a differentiable function on $[-\theta, \theta]$ and express the derivative $f'(x)$ for $x \in [-\theta, \theta]$ in terms of the sequence $\{a_n : n \geq 0\}$.

(iv) Show that $f'(x)$ is continuous for $x \in [-\theta, \theta]$.

(v) Show that f is Riemann Integrable on $[-\theta, \theta]$.

Part 2:

(3) Let $\{a_n : n \geq 1\}$ be a sequence such that $\{a_n \geq a_{n+1} > 0, \ \ \forall n \geq 0\}$.
Show that **[10]**

$$\sum_{n=1}^{\infty} a_n \text{ converges to a finite limit if and only if } \sum_{k=1}^{\infty} 5^k a_{5^k} \text{ converges to a finite limit.}$$

(4) Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a four times continuously differentiable function. We will use the notation $f^{(n)}$ to denote the n^{th} derivative of f. Using the familiar notation $g'(\alpha)$ for the derivative of g at α, we have $f'(x) = f^{(1)}(x)$ and for $n \geq 1$, $(f^{(n)})'(x) = f^{(n+1)}(x)$.

Let $\alpha \in \mathbb{R}$ be such that $f'(\alpha) = 0$, $f^{(2)}(\alpha) = 0$, $f^{(3)}(\alpha) = 0$ and $f^{(4)}(\alpha) > 0$. Show that α is a point of (local) minima of f: namely $\exists \delta > 0$ such that **[12]**

$$f(\alpha) \leq f(x) \text{ for } x \in (\alpha - \delta, \alpha + \delta).$$

(5) Let $f : \mathbb{R} \mapsto \mathbb{R}$ be defined by $f(x) = x^3 \sin^2(\frac{1}{x^2})$ for $x \neq 0$ and $f(0) = 0$.
Show that f is differentiable at all points $x \in \mathbb{R}$ and f' is bounded. Is f' continuous at $x = 0$? **[10+3+3]**

(6) Let $f : [a, b] \mapsto \mathbb{R}$ be a bounded function such that f is continuous on (a, b). Show that f is Riemann integrable on $[a, b]$. **[12]**