CMI Linear Algebra 2021 Midsem

09:00-12:00, Sunday, 31 October 2021

Important instructions: This is an open-book exam. Do not consult each other.

(1) Write clearly at the start of the answer book:
Algebra 1 Midsem
Your name and roll number.

(2) Number the pages. Submit a pdf version. Name the file:
YourFullName_RollNumber_LinearAlgebraMidsem.pdf

(3) Send by email to: RamadasExamCmi@gmail.com

(4) Answers to arrive by 12:30. Late submissions will be ignored.

`Maximum Marks 30/35`

(1) For d any non-negative integer, let $\mathcal{P}_d$ denote the vector space of polynomial functions $P : \mathbb{R} \to \mathbb{R}$, of degree $\leq d$. Thus, a function $P \in \mathcal{P}_d$ is of the form

$$P(x) = a_0 + a_1 x + \cdots + a_d x^d$$

where $a_0, \ldots, a_d$ are real numbers.

(a) [1] What is the dimension of $\mathcal{P}_d$?

(b) [8] Consider the map $\mathcal{P}_d \to \mathbb{R}^2$ given by

$$P \mapsto (P(0), P(0) + P'(0))$$

where P' denotes the derivative of P. Is this map surjective? Let K denote the kernel of this map. What is the dimension of K? Exhibit a subspace $\tilde{K} \subset \mathcal{P}_d$ such that $\mathcal{P}_d = K \oplus \tilde{K}$. Exhibit a basis for $\tilde{K}$.

(2) [4] Let V be a real vector space and suppose $dim\ V = 10$. Suppose given 2 linear maps λ_1, λ_2 from V to $\mathbb{R}$. Prove that the set

$$W = \{v \in V \big| \lambda_1(v) = 0,\ and\ \lambda_2(v) = 0\}$$

is a subspace of V. What are the maximum and minimum possible dimensions of W?

(3) Let V be a finite-dimensional real vector space. The set $\hat{V}$ of linear maps $\lambda : V \to \mathbb{R}$ is a vector space, with $\lambda_1 + \lambda_2$ defined by

$$(\lambda_1 + \lambda_2)(v) = \lambda_1(v) + \lambda_2(v),\ v \in V$$

and so on. (This vector space $\hat{V}$ is called the *dual* of V.)

(a) [2] Prove that $\hat{V}$ is also finite-dimensional, and in fact $dim\ V = dim\hat{V}$.

(b) [6] If $\hat{V}$ is a direct sum of two subspaces $\hat{V}_1$ and $\hat{V}_2$, prove that V is a direct sum of the subspaces

$$V_1 \equiv \{v \in V \big| \lambda(v) = 0\ \forall \lambda \in V_2\}$$
$$V_2 \equiv \{v \in V \big| \lambda(v) = 0\ \forall \lambda \in V_1\}$$

What are the dimensions of V_1, V_2 in terms of the dimensions of $\hat{V}_1$ and $\hat{V}_2$?

(4) [7] Consider the matrix

$$T \equiv \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Describe $ker(\hat{T})$ and $image(\hat{T})$, where $\hat{T} : \mathcal{M}_{7\times1} \to \mathcal{M}_{7\times1}$ is the map $\hat{T}(\vec{v}) = T\vec{v}$. ($\mathcal{M}_{7\times1}$ is the space of column vectors of length 7).

What is

$$T^7 \equiv \underbrace{T \circ \cdots \circ T}_{7\ times}\ ?$$

Hint: Consider the action of $\hat{T}$ on the vectors of the standard basis of $\mathcal{M}_{7\times1}$

(5) [7] Write the system of equations

$$\begin{aligned} v_1 &= v_2' \\ v_2 &= v_3' \\ &\cdots \\ v_9 &= v_{10}' \\ v_{10} &= v_1' \end{aligned}$$

in matrix form

$$A\vec{v} = \vec{v}'$$

where $\vec{v}$ and $\vec{v}'$ are column vectors. (In other words, find the matrix A.) What is the inverse matrix A^{-1}? What is the matrix

$$A^{10} \equiv \underbrace{A \circ \cdots \circ A}_{10\ times}$$