CMI Linear Algebra 2021 Endsem

14:00-17:00, Saturday 11 December 2021

Important instructions: This is an open-book exam. Do not consult each other.

(1) Write clearly at the start of the answer book:
Algebra 1 Endsem
Your name and roll number.

(2) Number the pages. Submit a pdf version. Name the file:
$YourFullName\_RollNumber\_LinearAlgebraEndsem.pdf$

(3) Send by email to: RamadasExamCmi@gmail.com

(4) Answers to arrive by 17:30. Late submissions will be ignored.

`Maximum Marks 40/42`

(1) `[3+1]` Let $n \geq 1$. Suppose given n column vectors $\vec{v}_1, \ldots, \vec{v}_n$ of length n. Prove that they form a basis for $\mathcal{M}_{n\times 1}$ iff

$$det\ [\vec{v}_1 \ldots \vec{v}_n] \neq 0$$

Here $[\vec{v}_1 \ldots \vec{v}_n]$ denotes the $n \times n$ matrix with j^{th}column equal to $\vec{v}_j$.

Formulate and prove an analogous result for *row* vectors $\vec{w}_1, \ldots, \vec{w}_n$.

(2) `[3]` Let N be a real 2×2 nilpotent matrix. If N is nonzero, prove that there is an invertible 2×2 matrix T such that

$$TNT^{-1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(3) Let C be the matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) `[1]` Describe the image of C.

(b) `[2]` Describe the kernel of C.

(4) [2] Let A and B be 100×100 matrices, with

$$A = \begin{bmatrix} 1 & 0 & . & . & 0 & 0 \\ 0 & 2 & . & . & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 99 & 0 \\ 0 & 0 & . & . & 0 & 100 \end{bmatrix}$$

If B is nilpotent and $AB = BA$, what can you say about B?

(5) Let I_2 be the 2×2 identity matrix. Let A be a real 2×2 matrix such that

- A is nonzero,
- A is not the identity matrix,
- $A^2 = A$

(a) [2] Is there an invertible matrix T such that $TA = I_2 T$?

(b) [2] is there a non-zero matrix T such that $TA = I_2 T$?

(6) [2] Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Prove or disprove: if $p_B(t) = t^2 - 2t + 1$, then B is similar to A.

(7) [3] If $\hat{N} : V \to V$ is a nilpotent linear map, with V finite-dimensional, prove that $I_V + \hat{N}$ is invertible.

(8) [3] Let V be an real finite-dimensional inner product space. Recall that an orthogonal transformation $\hat{O} : V \to V$ is one that preserves inner products. Prove that given such an $\hat{O}$ and a subspace $W \subset V$ invariant under $\hat{O}$, the orthogonal complement $W^{\perp}$ of W is also invariant under $\hat{O}$.

(9) [3] Fill in the blanks in the following (real) 2×2 matrix O so that it becomes orthogonal.

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & - \\ \frac{1}{\sqrt{2}} & - \end{bmatrix}$$

How many ways can you do this? What if you demand that $det\ O = 1$ as well?

(10) [5] For any integer $N > 0$, let V_N be the space of real polynomials of degree $\leq N$. Clearly

$$(P, Q) = \int_{-1}^{1} P(x)Q(x)dx$$

defines an inner product on V_N. The Gram-Schmidt process applied to the basis

$$1, x, x^2, x^3, \ldots, x^N$$

yields a sequence of polynomials $(p_0, p_1, p_2, \ldots, p_N)$ with the property that

$$\int_{-1}^{1} p_i(x) p_j(x) dx = \delta_{i,j}$$

Compute p_0, p_1, p_2, p_3 assuming $N \geq 3$. (Note that the actual value of N does not matter!)

Prove that p_{2m} is even and p_{2m+1} odd under $x \mapsto -x$. That is,

$$p_{2m}(-x) = p_{2m}(x) \quad p_{2m+1}(-x) = -p_{2m+1}(x)$$

(11) Let A be the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) [4] Find an orthogonal matrix S such *det* $S = 1$ and $S^{tr} A S$ is diagonal. Show your computations.

(b) [3] Find an orthogonal matrix T with *det* $T = -1$ such that $TA = AT$.

(c) [3] How many such T exist?