CHENNAI MATHEMATICAL INSTITUE

Probability Theory

Final Examnation Duration: 2hrs

Answer ALL queestions in Part A and any THREE questions in PART B.

PART A

Maximum mark for each quesion: 5

1. A biased coin is such that $P(H) = 1/5, P(T) = 4/5$. Find the least number n such that the probability that head appears at least once in n tosses is at least $1/2$.

2. Suppose that N number of fair dice are thrown where N is a random number. It is given that $P(N = k) = 1/2^k, k \geq 1$. Let S be the sum of the numbers scored in the throw. Find the probability that (i) $S = 4$, given that $N = 2$, (ii) $N = 2$ given that $S = 4$.

3. A particle at the t-th second moves one unit to the left or to the right with probabilities p or $1 - p$ respectively. At time $t = 0$ it is at the origin. Find the probability that it would be at the origin again (a) at time $t = 10$ seconds, (b) at time $t = 21$ seconds.

4. Suppose that $X = N(0, 1)$, the standard normal variate. Find the density function of $Z = X^2$. Find the expectation of Z.

5. Define the moment generating function $M_X(t)$ of a random variable X. (i) Find $M_X(t)$ when X is a Poisson variable with parameter λ, (ii) standard normal variate $N(0, 1)$.

6. Find the mean and variance of the random variable $Z = 2X - 3Y$ where X, Y is are independent random variables, given that X, Y have means $2, -2$ and variance $4, 5$ respectively.

Part B

Maximum marks for each question: 10.

7. Find the density function of the random variable $X + Y$ given that X, Y are independent random variables with density functions $f_X(x) = e^{-x}, x \geq 0$ and $f_Y(y) = 1, 0 \leq y \leq 1$.

8. Show that, if X and Y are independent exponential random variables with the same parameter λ, then $X + Y$ is also a random variable $\Gamma(\lambda, 2)$-distribution by (i) direct computation of the density function of f_{X+Y} as a convolution product, (ii) using characteristic functions. (*Use may use the fact that the random variable with* $\Gamma(\lambda, s)$*-distribution has characteristic function* $\phi(t) = (\frac{\lambda}{\lambda - it})^s$*.*)

9. An urn contains n balls numbered $1, 2, \ldots, n$. Fix a positive integer $k \leq n$. In an experiment k balls are drawn at random (without replacement) and their numbers are noted down. Let S be the random variable that calculates their sum. Show that the expectation of S equals $k(n+1)/2$.

10. Suppose that X, Y are exponential random variables with parameter λ, μ and are independent. Find the joint distribution function $g_{U,V}(u, v)$ of $U = X + Y$ and $V = Y^2$.

11. Suppose that X_n is a Poisson variable with parameter n and let $Y_n := (X - n)/\sqrt{n}$. Show that $\lim_{n\to\infty} \phi_n(t) \to \phi(t)$ where $\phi_n(t)$ is the characteristic funcion of Y_n and ϕ is the characteristic function of the standard normal variable. Deduce that $\lim_{n\to\infty} e^{-n} \sum_{0\leq k\leq n} n^k/k! \to 1/2$.

BEST WISHES