CHENNAI MATHEMATICAL INSTITUTE

Discrete Mathematics

End Semester Examination:
Date: July 28, 2021.

(1) Let $x_1, x_2, \dots, x_n$ be Boolean variables (i.e) they can take values in $\{0,1\}$. We denote the complement of a Boolean variable x_i by $\overline{x_i}$. So $\overline{x_i}$ takes the value 0 if and only if x_i takes the value 1. The collection of variables in $L = \{x_1, x_2, \dots, x_n, \overline{x_1}, \overline{x_2}, \dots, \overline{x_n}\}$ are called literals.
A k-CNF is an expression of the form

$$C_1 \wedge C_2 \wedge \cdots \wedge C_m$$

where each C_i is of the form $\ell_{i_1} \vee \ell_{i_2} \vee \cdots \vee \ell_{i_k}$, where $\ell_{i_j} \in L$ and $k \geq 2$. We say C_i is true if at least one literal in C_i has value 1. So for example when $k = 3$, and $C_i = x_3 \vee \overline{x_2} \vee x_5$. C_i is not true precisely when x_3 is 0, x_2 is 1 and x_5 is 0. Show using the probabilistic method that there is an assignment of values to the variables $x_1, \dots, x_n$ such that at least half of the C_i's are true. 6 marks

(2) Consider the following partial order on k element subsets of $\{1, 2, \dots, n\}$. We write each k-element subset in set notation with it's elements increasing from left to right. We say $\{a_1, a_2, \dots, a_k\} < \{b_1, b_2, \dots, b_k\}$ iff $a_k < b_k$, or $a_k = b_k$ and $a_{k-1} < b_{k-1}$, or $a_k = b_k$, $a_{k-1} = b_{k-1}$ and $a_{k-2} < b_{k-2}$, and so on. Write down the partial order for three element subsets of $\{1, 2, 3, 4, 5\}$. How many elements precede a given a subset $\{a_1, a_2, \dots, a_k\}$? Verify that your formula works for the subset $\{1, 3, 4\}$ of $\{1, 2, 3, 4, 5\}$. Show that for any integer m between 0 and $\binom{n}{k} - 1$, there exist unique integers $0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_k \leq n-1$, such that

$$m = \binom{\alpha_k}{k} + \binom{\alpha_{k-1}}{k-1} + \ldots \binom{\alpha_1}{1}.$$

Use the fact that the number of elements preceeding an element in the given order is a number between 0 and $\binom{n}{k} - 1$. 8 marks

(3) For a directed graph G we say subgraph T is a directed tree rooted at vertex v if, in T, v has no incoming edges, and for each vertex u of G there is precisely one directed path from v to u in the T - so the subgraph looks liked a tree rooted at v.
Let G be a directed acyclic graph with vertex r of indegree 0. Suppose G has a directed tree rooted at r. Let the indegree of each vertex v in G be $d(v)$. Show that the number of directed trees rooted at v is $\Pi_{v \neq r} d(v)$. Make sure you prove that your count is correct, show that it counts precisely trees rooted at r and exhausts all trees rooted at r. 6 marks

(4) Let S be a subset of vertices of a connected graph G and let $o(G-S)$ denote the number of odd components in $G-S$. Let $d(S)$ be defined as $o(G-S) - |S|$. Show that the largest matching in the graph is half of $min_{S \subseteq V}\{n - d(S) : S \subseteq V\}$. 6 marks

(5) Let G be a graph on n vertices having m edges. Show that G has at least $m - n + 1$ cycles. 4 marks

(6) Let G be a simple connected graph and let $X \subseteq E$ be a minimum subset of edges with the property that every vertex is incident on one edge in X. We call X a minimum edge cover. Show that the connected components of the subgraph induced by X are all stars. Recall that a star is a connected graph containing a vertex which is connected to all the other vertices in the graph of degree one. Using this show that the size of the largest matching in G plus the size of a minimum edge cover is equal to the number of vertices in G. 8 marks

(7) Let G be a graph without a perfect matching. Show that there is a vertex in G such that every edge incident on that vertex belongs to some maximum cardinality matching of G. 6 marks

(8) Show that a tree has strictly more leaves than the number of vertices of degree three or more. Can we relax the condition on the degree? 6 marks