Analysis II
Final exam
22/07/21, 10:00-14:00

You may use your class notes and textbook during the exam. No other sources are permitted. In a multi-part question, the result of one part may be used in a succeeding part even if you have not been able to derive it.

1. (*5×4 = 20 points*) True or false? Either give a proof or a counterexample.
   (a) Let $\mathcal{C} \subset \mathbb{R}^1$ denote the Cantor set, and let $\mathcal{D}$ be the subset $(\{0\} \times \mathcal{C}) \cup (\{1\} \times \mathcal{C})$ of $\mathbb{R}^2$ with the induced metric. Then, $\mathcal{C}$ is homeomorphic to $\mathcal{D}$.
   (b) Let B be an open ball in $\mathbb{R}^2$, and $f : B \to \mathbb{R}^1$ a continuous function all of whose directional derivatives are defined and *vanishing* on B. Then, f is constant even if it is not differentiable.
   (c) An uncountable subset of $\mathbb{R}^1$ must have a limit point.
   (d) Let function $f : I \to \mathbb{R}^1$ be a function, where $I \subset \mathbb{R}^1$ is an interval. If the graph[1] of f is connected, then f is continuous.

2. (*4+3+3=10 points*) Define f is $\mathbb{R}^2$ by
$$f(x, y) = 2x^3 + 6xy^2 - 3x^2 + 3y^2.$$
   (a) Find the points where the gradient of f vanishes. Which of these are local maxima and which are local minima?
   (b) Let $S = \{(x, y) : f(x, y) = 0\}$. Find those points of S which have no neighbourhoods in which the equation $f(x, y) = 0$ can be solved for one variable in terms of the other.
   (c) Show that there exists a differentiable function g defined a neighbourhood of 0 such that $g(0) = \frac{3}{2}$ and $f(g(y), y) = 0$. Find $g'(0)$.

3. (*10 points*) If a function $f : X \to Y$ is a function between metric spaces, the *graph* of f is the set of points $(x, f(x))$ in $X \times Y$, for $x \in X$.

   If X is a compact subset of $\mathbb{R}^1$ and f is a real-valued function on X, show that f is continuous on X if and only if the graph of f is compact in $\mathbb{R}^2$.

4. (*10+10=20 points*) Let $\mathcal{C}[0, 1]$ denote the *set* of continuous functions on the interval $[0, 1]$.
   (a) Show that the function $d_1(f, g) := \int_0^1 |f(t) - g(t)|dt$ defines as metric on $\mathcal{C}[0, 1]$. Is the metric space $(\mathcal{C}[0, 1], d_1)$ complete?
   (b) Define $\phi : \mathcal{C}[0, 1] \to \mathcal{C}[0, 1]$ by
$$\phi(f)(x) = \int_0^x t(t + f(t))dt.$$
   Show that ϕ is a contraction mapping for the usual sup metric. Use ϕ to find an approximate solution y to the differential equation
$$\frac{dy}{dx} = x(x + y), \qquad y(0) = 0, \qquad (0 \leq x \leq 1),$$
   accurate to one decimal place. Start with the approximation $y_0 = 0$.

5. (*7+3=10 points*) Let a, b, c, p, q be real nonnegative constants with $p, q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

[1] See problem 3.

(a) Maximize the function $f(x, y) = ax + by$ subject to the constraints $g(x, y) = x^q + y^q = c$, $x, y \geq 0$.

(b) Deduce Hölder's inequality:

$$ax + by \leq (a^p + b^p)^{\frac{1}{p}} \times (x^q + y^q)^{\frac{1}{q}}.$$

6. (*1+3+3+3=10 points*) Let $B = \{x \in \mathbb{R}^n : |x| < 1\}$ be the open unit ball, and $f : B \to \mathbb{R}^n$ a C^1-function, which extends to a *continuous* function on the closed unit ball $\overline{B}$. Suppose that the Jacobian of f is positive on B. Suppose further that $f(x) = x$ on the boundary of B, that is, whenever $|x| = 1$.

   (a) Show that if X is an open subset of B, then $f(X)$ is open in $\mathbb{R}^n$.

   (b) Show that $f(\overline{B})$ is contained in $\overline{B}$ (where is the latter $\overline{B}$ is a subset of the range).

   (c) Show that $f(B) = B$.

   (d) Show that if $y \in f(B)$, then $f^{-1}(y)$ is a finite subset of B.