Algebra 2
Final exam
July 26th, 2021
2 pm to 4:30 pm

Instructions

- The solutions are to be uploaded, as a single scanned file, on Moodle before 5 pm.
- In case of difficulties uploading the solutions contact us immediately.
- Any result that we did in the course could be directly used (after quoting it).
- Results that we didn't encounter in the course should be proved before using.
- All cyclic groups are equipped with multiplication and denoted as $C_n = \langle a \mid a^n = 1 \rangle$.

Question 1
In each of the following problem generators and relations of some finite group are given. Determine the order of the group in each case. Provide short but sufficient reasoning for your answer. [6 points]

(a) $G = \langle x, y \mid x^3 = y^2 = (xy)^2 = 1 \rangle$.

(b) $G = \langle x, y \mid x^3 = y^2 = (xy)^3 = 1 \rangle$.

(c) $G = \langle x, y \mid xy^2 = y^3x, yx^3 = x^2y \rangle$.

(Hint: either write down the elements in terms of x, y or just exhibit an isomorphism with a known group.)

Question 2
In each of the following problem two groups are given. Determine whether or not they are isomorphic. Provide short but sufficient justification. [6 points]

(a) $G = C_9 \times C_9$ and $H = C_{27} \times C_3$.

(b) $G = \mathrm{Aut}(C_{16})$ and $H = C_2 \times C_4$.

(c) $G = D_6$ (the dihedral group of order 12) and $H = D_3 \times C_2$.

Question 3
Answer the following (short) questions. [6 points]

(a) How many elements of order 6 are there in $C_9 \times C_6$?

(b) How many elements of order 25 are there in $C_5 \times C_{25}$?

(c) What are possible orders of elements in the group $S_3 \times C_5$?

Question 4
Answer the following. [6 points]

(a) List all abelian groups (up to isomorphism) of order 160.

(b) Prepare a table listing elements of order $1, 2, 4, 8, 16$ in every abelian group of order 16. The rows of this table are indexed by groups (up to isomorphism) and the columns by possible orders.

(c) Suppose G is an abelian group of order 168 and that G has exactly 3 elements of order 2. Determine the isomorphism class of G.

Question 5
Answer the following. [6 points]

(a) Suppose $\phi : C_{50} \to C_{15}$ is a homomorphism mapping a^7 to b^6. Where a, b are generators of C_{50}, C_{15} respectively. Determine a formula for $\phi(a^k)$ and obtain explicit descriptions of $\ker(\phi), \mathrm{im}(\phi)$ and $\phi^{-1}(b^3)$.

(b) Show that there doesn't exist a surjective homomorphism from $C_8 \times C_2 \to C_4 \times C_4$.

Question 6
Answer the following. [8 points]

(a) Show that there can't be a simple group of order 56.

(b) Show that there are exactly two groups (up to isomorphism) of order 45.

Question 7
We say that a positive integer $n > 1$ is **prime-like** if: [12 points]

- n is square free (i.e., $p^k \mid n \Rightarrow k = 1$) and,
- if p_i, p_j are two distinct prime factors of n then $p_i \nmid (p_j - 1)$ (and $p_j \nmid (p_i - 1)$).

Prove that if there is a unique group (up to isomorphism) of order n then n is prime-like. Conversely, prove that if $n = p_1p_2p_3$ is prime-like (with 3 distinct prime factors) then there is a unique group of order n. (Hint: For the first implication you may want to try contradiction. For the second implication try induction and the fact that groups of odd order are solvable.)