Analysis 1 - End Sem

Date & Time: 22 March, 2021; 14:00 to 16:30 (Upload before 17:00) **Maximum marks:** 100

Instructions:

- When you use any result **proved in the class,** do state it clearly.
- If you use a result **not proved in the class**, do prove it.
- Do make sure that you finish well in time so that you have time to upload.
- Please respect the integrity and sanctity of the examination process.

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1. (20 marks)

   (a) Let $\{a_n\}$ be an unbounded sequence of real numbers. Does it necessarily have a monotonic (increasing or decreasing) subsequence? Justify.

   (b) Let $\{a_n\}$ and $\{b_n\}$ be two Cauchy sequences in $\mathbb{Q}$. Then is the sequence $\{c_n\}$ given by $c_n := |a_n - b_n|$ Cauchy? Justify.

2. (20 marks)

   Let $\{x_n\}$ be a sequence of real numbers. Prove or disprove the following statements.

   a) If the series $\sum_{n=1}^{\infty} x_n$ converges in $\mathbb{R}$, so does the series $\sum_{n=1}^{\infty} x_n^2$.
   b) If the series $\sum_{n=1}^{\infty} x_n$ converges in $\mathbb{R}$, so does the sequence (y_n) where $y_n := \sum_{j=n}^{\infty} x_j$

3. (20 Marks)

   Recall the exponential function $e^x : \mathbb{R} \to (0, \infty)$ which we showed to be bijective. Let log denote its inverse function. Show that log is continuously differentiable. Also show that for any $x \in [1, 2]$, we have

   $$\log(x) = \int_1^x \frac{1}{t}.$$

4. (16 Marks)

   Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and bijective. Then show that the inverse f^{-1} is also continuous.
   Hint: Intermediate Value Theorem

5. (24 marks)

   Let $f : [0, 1] \to \{0, 1\}$ be defined as $f(x) = 0$ if and only if $x = 0$.

   Let $g : [0, 1] \to [0, 1]$ be defined as $g(x) = 0$ if $x = 0$ or $x \notin \mathbb{Q}$.

   Further $g(x) = \frac{1}{q}$ when $x = \frac{p}{q}$ (p, q are co-prime).

   Prove or disprove the following statements.

   a) f is integrable
   b) g is integrable
   c) The composition fog is integrable.