CMI ALGEBRA 1 MIDSEM
13 FEBRUARY 2021, 10:30-13:30

Important instructions: This is a **closed-book** exam. **Do not** consult each other. Write clearly at the start of the answer book: "Algebra 1 Midsem. Your name and roll number". Number the pages. Submit a pdf version.

Name the file *YourFullName_RollNo_Algebra1Midsem.pdf*

Send by email to: RamadasExamCmi@gmail.com

Answers to arrive by 14:00. Late submissions will be ignored.

`[Maximum Marks: 30]`

(1) `[4 marks]` *Without using matrices*, exhibit a linear map $\hat{N} : V \to V$ on a 7-dimensional vector space V such that the vector spaces in the following sequence

$$V \supset \hat{N}(V) \supset \hat{N}^2(V) \supset \hat{N}^3(V) \supset \hat{N}^4(V)$$

have dimensions $7, 5, 3, 1, 0$.

(2) `[2+2 marks]` Let $\hat{A} : V \to V$ be a linear map, with V an n-dimensional complex vector space.

(a) If the characteristic polynomial $p_{\hat{A}}(t)$ has a root λ with multiplicity 2, then $\hat{A}$ has at least two linearly independent eigenvectors with eigenvalue λ. True or false?

(b) if $\hat{A}$ has two linearly independent eigenvectors with eigenvalue λ, then λ is a root of $p_{\hat{A}}(t)$ and the multiplicity of λ is at least two. True or false?

Give short explanations for your choices.

(3) `5 marks]` Let n be a natural number, and μ_n the set of n^{th} roots of unity:

$$\mu_n = \{1, \xi, \xi^2, \ldots, \xi^{n-1}\}, \ where\ \xi = e^{i\frac{2\pi}{n}}$$

Let V be the complex vector space of complex valued functions $f : \mu_n \to \mathbb{C}$. Let $\hat{T} : V \to V$ be the map $\hat{T} f(\xi^l) = f(\xi^{l+1})$. Find the eigenvalues of $\hat{T}$. Compute $det\ \hat{T}$.

(4) Let V be a n-dimensional complex vector space, and $\hat{A} : V \to V$ a linear map. Suppose that

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(a) There exists an integer $m > 0$ such that $\hat{A}^m(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$, and

(b) $p_{\hat{A}}(t) = (t - \xi)^n$.

`[7 marks]` Prove that $\hat{A} = \xi I_V$. (*You can try the simpler case* $n = 2$ *and* $\xi = 1$ *for a maximum of* `[4 marks]`.)

`[1 mark]` What can you say about ξ?

(5) For any $n \times n$ matrix A, let $\mathcal{A} : \mathcal{M}_{n\times n} \to \mathcal{M}_{n\times n}$ be the linear map

$$\mathcal{A}(C) = AC - CA$$

(a) `[1 mark]` Prove that $det\ \mathcal{A} = 0$.

(b) Recall that the *trace* of a square matrix is the sum of its diagonal entries. If this is zero, we say that the matrix is *traceless*. Let $\mathcal{M}^0_{2\times 2} \subset \mathcal{M}_{2\times 2}$ be the subspace of traceless 2×2 matrices.

– `[1 mark]` What is $dim\ \mathcal{M}^0_{2\times 2}$?

– `[1 mark]` Show that for any 2×2 matrix B, the image of the corresponding map $\mathcal{B} : \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ is contained in $\mathcal{M}^0_{2\times 2}$. (To be explicit: $\mathcal{B}(C) = BC - CB$.)

– `[3 marks]` By the previous question, for any B, the restriction of $\mathcal{B}$ to $\mathcal{M}^0_{2\times 2}$ gives a linear map $\mathcal{B}|_{\mathcal{M}^0_{2\times 2}} : \mathcal{M}^0_{2\times 2} \to \mathcal{M}^0_{2\times 2}$. Let B_1 denote the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Compute $det\ \mathcal{B}_1|_{\mathcal{M}^0_{2\times 2}}$.

(c) `[3 marks]` If A_1 is the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

determine the eigenvalues of the corresponding map $\mathcal{A}_1 : \mathcal{M}_{3\times 3} \to \mathcal{M}_{3\times 3}$. (*This problem might need some computations; perhaps you could do this at the end.*)