Chennai Mathematical Institute

Topology

Quiz-3

Duration: 1 hour | Date: Friday, 12th April, 2024.

- **Answer any THREE questions. Give brief answers.**

Q1. (a) Let $r : X \to A$ be a retraction. Show that $r_* : \pi_1(X, a) \to \pi_1(A, a)$ is a ~~mono~~epimorphism of groups.

(b) Show that there is no retraction $\mathbb{D}^2 \to \mathbb{S}^1$ where $\mathbb{D}^2 \subset \mathbb{R}^2$ denotes the closed unit disk with centre the origin.

Q2. (a) Show that $\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ defined as $\alpha(x, y) = (-x, -y)$ is homotopic to the identity map.

(b) Let $\rho : \mathbb{S}^1 \to \mathbb{S}^1$ be the reflection map $(x, y) \to (x, -y)$. Determine the induced homomorphism $\pi_1(\mathbb{S}^1, 1) \to \pi_1(\mathbb{S}^1, 1)$.

(c) Is ρ homotopic to the identity?

Q3. Let $p : (\widetilde{X}, \tilde{x}_0) \to (T, e)$ be a covering projection where $T = \mathbb{S}^1 \times \mathbb{S}^1$ and $e = (1, 1)$. Assume that with $\widetilde{X}$ connected. Suppose that $p_*(\pi_1(\widetilde{X}, \tilde{x}_0)) = \mathbb{Z}(4e_1 + 6e_2) + \mathbb{Z}(12e_1 + 36e_2) \subset \mathbb{Z}^2 = \pi_1(T, 1)$. (a) Determine the group $Deck(p)$.

(b) Is the lift $\tilde{\sigma} : I \to \widetilde{X}$ over p of the loop $\sigma : [0, 1] \to T$, defined as $t \mapsto (1, \exp(2\pi i 18t))$, starting at $\tilde{x}_0$ a closed loop?

Q4. (a) Determine the fundamental group of the following spaces:
$X = \mathbb{R}^2 \setminus S$ where $S = I \times \{0\}$ and $Y = \mathbb{R}^3 \setminus L$ where L is a straight line.

(b) Let $Z = \bigcup_{n \in \mathbb{N}} C_n$ where $C_n \subset \mathbb{R}^2$ is the (boundary of the) triangle with vertices $O = (0, 0), A_n = (-n, n), B_n = (n, n)$. Show that $U = Z \cap B$ is contractible where $B = B_1(O)$ is the open unit ball centred at O..