CHENNAI MATHEMATICAL INSTITUE

Topology– Mid-Semester examination

Duration: $2\frac{1}{2}$ hours | Date: 1st Mar, 2024.

∘ **Answer any FIVE questions.** ∘ **Give brief answers.**

Q1. (a) Show that $\mathbb{Q}$ is dense in $\mathbb{R}_\ell$ (the reals under lower limit topology).
(b) Show that if X is a compact metric space, then X has a countable basis for its topology.
(c) Show that $X = S_\Omega \cup \{\Omega\}$ (where $\alpha < \Omega\ \forall \alpha \in S_\Omega$) under order topology is compact but does not have a countable basis.

Q2. (a) Show that $\mathbb{R}^\omega$ (with product topology) is metrizable.
(b) Consider $X = I^\omega$ under the uniform topology. Find the limit points of the set $S = \{0,1\}^\omega \subset X$.
(c) Show that X is not compact.

Q3. Let X, Y be metric spaces.
(a) Define the notion of uniform convergence of a sequence of maps $(f_n : X \to Y)_{n\geq 1}$.
(b) For $n \geq 1$, let $f_n : [0,1) \to I$ be the function $f_n(t) = t^n\ \forall t$. Show that $\lim_{n\to\infty} f_n(t) = 0\ \forall t \in [0,1)$ but that the convergence is not uniform.

Q4. (a) Show that the space $X = \mathbb{N} \times [0,1)$ in the dictionary order topology is path connected.
(b) Show that the space $[0,1]\times[0,1)$ is locally compact under the dictionary order topology.
(c) Show that S_Ω is locally compact but not compact.

Q5. (a) Suppose that $S \subset \mathbb{R}$ is a nontrvial subgroup (under addition) and that it is discrete (in the subspace topology). Show that S is closed and that $S \cong \mathbb{Z}$. (as groups)
(b) Show that any cyclic subgroup of $\mathbb{S}^1$ is either finite or dense in $\mathbb{S}^1$.

Q6. (a) Suppose that Y is the one-point compactification of the unit ball $B = \{v \in \mathbb{R}^2 \mid ||v|| < 1\}$. Show that $Y \cong \mathbb{S}^2$.
(b) Let X be a locally compact Hausdorff, non-compact topological space and let Y be its one-point compactification. A sequence $(x_n)_{n\geq 1}$ in X is said to *diverge*, written $\lim x_n \to \infty$, if, given any compact subset $K \subset X$, there exists an $N \geq 1$ such that $x_n \notin K$ for all $n \geq N$. Show that a continuous function $f : X \to \mathbb{R}$ extneds to a continuous function $\tilde{f} : Y \to \mathbb{R}$ if, whenever $\lim x_n \to \infty$, we have $(f(x_n))_{n\geq 1}$ converges to a point of $\mathbb{R}$.