Chennai Mathematical Institue

Topology– End-Semester examination

Date:20th April, 2024. *Duration 3hours.*

- Answer any FOUR questions from Part A and FOUR from Part B.
- Each question in Part A carries **six** marks and each in Part B carries **eight** marks.
- *Give brief answers.*

Part A

Q1. Define a linear continuum and determine which of the following spaces with order topology are linear continua: (a) $J = [0,1] \cup (2,3]$ with the usual order, (b) $I \times I$ with the dictionary order.

Q2. Define the notion of local compactness. (a) Give an example of a bounded subspace of $\mathbb{R}$ which is *not* locally compact. (b) Suppose that X is locally compact Hausdorff and $Y \subset X$ is a closed subset, show that Y is locally compact.

Q3. Define the notion of a completely regular space. Prove or disprove: "*Any subspace of a completely regular space is completely regular.*"

Q4. Let $X = \prod_{n\in\mathbb{N}} X_n$ with product topology. Prove: (a) If $X_n = \{1, 2, \ldots, 10^n\}$ $\forall n$, then X is totally disconnected. (b) if $X_n = [0, 10^{-n})$ for all $n \geq 1$, then X is not locally compact.

Q5. (a) Suppose that X has a countable dense subset. Show that if $\{U_\alpha\}_{\alpha\in J}$ is a collection of pairwise disjoint open subsets, then J is countable.
(b) Give an example of a compact Hausdorff space which does not have a countable dense subset.

Q6. (a) Let $X = \mathbb{R}^2 \setminus K$, with subpace topology, where $K = C \times C$ where $C \subset \mathbb{R}$ is the Cantor set. Show that X is path connected.
(b) Prove that any open subset X of $\mathbb{R}^n$ is locally contractible, i.e., every point $p \in X$ has an open neighbourhood V which is contractible.

Part B

Q7. (a) Let $X, Y = \mathbb{R}^\omega$ where X is given product topology and Y the uniform topology. Let $a(k) = (a_n(k))_{n\geq 1} \in I^\omega$ where

$$a_n(k) = \begin{cases} 1/n, & \text{if } n \neq k, \\ 1, & \text{if } n = k. \end{cases}$$

Determine in which of the spaces X, Y, the sequence $(a(k))_{k\geq 1}$ is convergent.
(b) Show that $x = (x_n)_{n\geq 1} \in Y$ belong to the same connected component of Y as $0 = (0)_{n\geq 1} \in Y$ if x is a bounded sequence.

Q8. (a) Show that there is a continuous surjection $\beta(\mathbb{N}) \to \beta(\mathbb{N}^\omega)$.
(b) Let S be the topologist's sine curve $S = \{(t, \sin(\pi/t)) \in \mathbb{R}^2 \mid 0 < t \leq 1\}$ and let $\overline{S}$ be the closure of S in $\mathbb{R}^2$. Let $\alpha : (0,1] \to \overline{S}$ be the embedding $t \mapsto (t, \sin(\pi/t))$. Find a

1

continuous function $f : \overline{S} \to \mathbb{R}$ such that $f(\alpha(t)) = \sin(\pi/t)$.
(c) Suppose that $h : (0, 1] \to [-1, 1]$ is a continuous function such that

$$h(1/n) = \begin{cases} 1, & \text{if } n \equiv 0 \mod 2, \\ -1, & \text{if } n \equiv 1 \mod 2. \end{cases}$$

Show that there does not exist any continuous function $g : \overline{S} \to \mathbb{R}$ such that $g(\alpha(t)) = h(t)$ for all $0 < t \leq 1$.

Q9. (a) Define the notion of a subspace $A \subset X$ being a strong deformation retract.
(b) Show that $\mathbb{S}^{n-1}$ is a strong deformation retract of $\mathbb{R}^n \setminus \{0\}$.
(c) Show that $S = C_0 \cup C_1$ is <u>not</u> a retract of $\mathbb{R}^2 \setminus \{0\}$ where C_j is the circles with centre at $(2j, 0)$ and unit radius.

Q10. Let $X \subset \mathbb{R}^2$ be the union of the following subsets $A = \mathbb{R} \times \{0\}, B_n, n \in \mathbb{Z}$, is the circle with unit radius and centre $4n, 1)$. (a) Define a covering projection $X \to \mathbb{S}^1 \vee \mathbb{S}^1$ where $p(B_0) = \mathbb{S}^1 \times \{1\}$. (b) Show that $p : X \to \mathbb{S}^1 \vee \mathbb{S}^1$ is a regular covering. (c) Determine the subgroup $p_*(\pi_1(X, \tilde{x_0})) \subset \pi_1(\mathbb{S}^1 \vee \mathbb{S}^1, (1, 1)) = F_2 = \langle a, b \rangle$ where $\tilde{x}_0 = 0 \in X$. (d) Describe the deck transformation group $Deck(p)$.

Q11. Let T be the torus $T = \mathbb{S}^1 \times \mathbb{S}^1$ and let $\alpha : T \to T$ be the map $(z, w) \mapsto (-z, w^{-1})$ $\forall z, w \in \mathbb{S}^1$. (a) Show that $\alpha(z, w) \neq (z, w)$ $\forall (z, w) \in T$, and $\alpha \circ \alpha = id$. Let $S = T/\sim$ where $(z, w) \sim \alpha(z, w)$. Show that the quotient map $q : T \to S$ is a covering projection.
(b) Using the usual identification of T as the quotient space $I^2/\sim$, show that $\alpha([s, t]) = (1/2 + s, 1 - t)$. Hence or otherwise, show that S is the quotient space of $R = [0, 1/2] \times I$ with quotient topology wiith identifications on are on only the boundary of the rectangle $\partial R = \{0, 1/2\} \times [0, 1] \cup [0, 1/2] \times \{0, 1\}$ as follows: $(0, t) \sim (1/2, 1 - t), \forall t \in I$ and $(s, 0) \sim (s, 1)$, $\forall s \in [0, 1/2]$. (Draw the diagram depicting the identifications.

Q12. Let G be the group with presentation $\langle x, y \mid xyx^{-1}y \rangle$. (a) Let $H \subset G$ be the subgroup generated by x^2, y. Show that H is a normal subgroup of G of index 2. (b) Show that x has infinite order and x^2 belongs to the centre of G.