Theory of Computation
Midsem Exam

October 4, 2023
Time: 2pm to 4pm
Total marks: 100

This is a two hour exam. Write clearly and precisely. In question 8 you can choose between 8(a) and 8(b).

(1) Let L be any regular language. Consider $L' = \{w \in L \mid \text{no proper prefix of } w \text{ is in } L\}$. Prove that L' is regular. **10 marks**

(2) Let L and L' be regular languages. Prove that their *perfect shuffle* $L'' = \{w = a_1b_1a_2b_2 \cdots a_kb_k \mid k \geq 0 \text{ where } a_1a_2 \cdots a_k \in L \text{ and } b_1b_2 \cdots b_k \in L'\}$ is regular. **10 marks**

(3) Give a self-contained proof (from first principles) that $L = \{a^ib^jc^k \mid i,j,k \geq 0 \text{ such that if } i = 1 \text{ then } j = k\}$ is not a regular language. **15 marks**

(4) Consider the context-free grammar $E \to E+E \mid E*E \mid (E) \mid x \mid y \mid z$, with E as start symbol and terminals as $T = \{(,),x,y,z\}$. Give a word in T^* generated by the grammar with two different derivation trees. **10 marks**

(5) Let $L = \{a^nb^n \mid n \geq 0\}$. Prove that its complement $\overline{L} = \{a,b\}^* \setminus L$ is a context-free language. **10 marks**

(6) Prove by induction that the context-free grammar consisting of productions $S \to \epsilon$, $S \to bA$, $S \to aB$, $A \to aS$, $A \to bAA$, $A \to a$, $B \to bS$, $B \to aBB$, $B \to b$, generates all words with an equal number of a's and b's. **10 marks**

(7) Is the language over $\{a,b,c\}$ consisting of all words with an equal number of a's, b's, and c's a context-free language? Justify answer with proof. **10 marks**

(8) Answer one of the following:

(a) Let L and L' be regular languages over alphabet $\{a,b\}$. Let $L'' = \{w \in L \mid \text{some } y \in L' \text{ has the same number of } a\text{'s as } w\}$. Is L'' regular? If yes, give a DFA or NFA for it. Else prove it is not regular.

or

(b) Let $L \subseteq \{a,b\}^*$ be any regular language. Let L' be the context-free language consisting of all words $w \in \{a,b\}^*$ with an equal number of a's and b's. Is the language $L'' =$

$\{w \mid ww' \in L \text{ for some } w' \in L'\}$ regular? If yes, give a DFA or NFA for it. Else prove it is not regular.

10 marks

(9) A linear context-free grammar over alphabet Σ has only productions of the form $A \to \alpha B \beta \mid \alpha$ for $\alpha, \beta \in \Sigma^*$, where A and B are variables in the grammar. In other words, the RHS of every production has at most one variable.

(a) Let L be a context-free language with a linear context-free grammar. Show that there is a constant n such that all words $z \in L$ of length at least n can be written as $z = uvwxy$ where $|uvxy| \leq n$, $|vx| \geq 1$, and $uv^iwx^iy \in L$ for all $i \geq 0$.

(b) Consider the context-free language $L = \{w \mid w \text{ has an equal number of } a\text{'s and } b\text{'s}\}$. Show using part (a) or otherwise that L cannot be generated by a linear context-free grammar.

10+5 marks