Theory of Computation
Final Exam

November 29, 2023
Time: 9.30am to 12.30pm
Total marks: 100

Write clearly and precisely.

(1)(a) Define recursive languages and recursively enumerable languages.

(b) State Rice's theorem for recursive index sets and for recursively enumerable index sets.

(c) Show that the language $L = \{\langle M \rangle \mid L(M) \text{ is not recursive}\}$ is undecidable.

(d) Is L recursively enumerable? Justify.

You may give direct proofs or apply Rice's theorem(s) with precise statements. **2+3+3+4 marks**

(2)(a) Is the language $L = \{\langle M \rangle \mid L(M) \cap L_u \neq \emptyset\}$ recursively enumerable?

(b) Is $\overline{L}$ recursively enumerable?

Justify answers. **4+4 marks**

(3)(a) What is the Post Correspondence Problem (PCP for short)?

(b) Show that checking if a context-free grammar is ambiguous is undecidable using PCP.

(c) Is the set $L = \{G \mid G \text{ is a CFG such that } \not{L}(G) \text{ is ambiguous}\}$ recursively enumerable?

2+4+4 marks

(4) Let $\Sigma = \{0, 1\}$ and $L \subseteq \Sigma^*$.

(a) If both L and $\overline{L}$ are recursively enumerable show that L is recursive.

(b) Suppose L is recursively enumerable. Show that there is a *recursive language* $A \subset \Sigma^* \times \Sigma^*$ such that

$$L = \{x \in \Sigma^* \mid \exists y \in \Sigma^* \ : \ (x, y) \in A\}.$$

4+6 marks

(5)(a) Let $L_1, L_2 \subseteq \Sigma^*$ be languages. When is L_1 said to be *many-one* reducible to L_2? In the context of oracle Turing machines, when is L_1 said to be *recursive in* L_2? When are L_1 and L_2 said to be (recursively) *equivalent*?

(b) Let $L = \{\langle M \rangle \mid L(M) \text{ is finite}\}$. Recall that $S_1 = \{\langle M \rangle \mid L(M) = \emptyset\}$. Show that there is no algorithm using S_1 as oracle for testing membership in L.

(c) Give an oracle algorithm, with L as oracle, for testing membership in S_1.
Justify answers. **5+5+5 marks**

(6)(a) State the recursion theorem.

(b) Let $\tau : \mathbb{N} \to \mathbb{N}$ be any *surjective* total recursive function. Using the recursion theorem show that for some positive integer i the two Turing machines $M_{\tau(i)}$ and $M_{\tau(i+1)}$ compute the same 1-ary partial recursive function.

(c) Show using the recursion theorem that any total recursive function $\sigma : \mathbb{N} \to \mathbb{N}$ must have infinitely many fixed points. I.e. there are infinitely many indices i such that $M_{\sigma(i)}$ and M_i compute the same 1-ary partial recursive function.

3+3+4 marks

(7) Let G be a finite group with identity element denoted by 1. Let G^* denote the set of all words over G. Consider the language $L = \{w \in G^* \mid w \text{ evaluates to } 1\}$.

(a) Show that L is regular.

(b) What is the minimum size of a DFA that accepts L?

Justify answers. **5+5 marks**

(8)(a) Give a deterministic pushdown automaton that accepts by final state for the language over $\{0, 1\}$ consisting of all strings with an equal number of 0's and 1's.

(b) Is there a nondeterministic PDA for it that accepts by empty stack?

(c) Is there a deterministic PDA for it that accepts by empty stack?

Justify answers with an explanation. **3+3+4 marks**

(9) Let Σ and Δ be two finite alphabets. Consider a substitution map $f : \Sigma \to 2^{\Delta^*}$ such that for each $a \in \Sigma$ its image $f(a) = R_a \subseteq \Delta^*$ is a regular language.

(a) For a language $L \subseteq \Sigma^*$ how is its image $f(L) \subseteq \Delta^*$ defined, obtained by applying the substitution map to L?

(b) For a language $L' \subseteq \Delta^*$ how its inverse image $f^{-1}(L') \subseteq \Sigma^*$ defined?

(c) If $L \subseteq \Sigma^*$ is regular is $f(L)$ regular?

(d) If $L' \subseteq \Delta^*$ is regular is $f^{-1}(L')$ regular?

Justify answers. **2+3+5+5 marks**