Theory of Computation

Assignment-4

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① Given a pair $\langle M, w \rangle$ ~~with~~ as input, where $\langle M \rangle$ is a TM code and $w \in \{0,1\}^*$ is its input. We shall prove that the problem of checking if M will halt ~~on~~ with a blank tape is an undecidable problem as follows:

Assume, to the contrary, that the given problem is decidable. Let R be the TM that solves the problem. Using this TM R, we can solve the original halting problem, which is to check if a TM M halts on a string $w \in \{0,1\}^*$, as follows:

Let H be the TM for solving the original ~~=~~ halting problem. Let M_w be the TM that implements M with modification as follows:

> Whenever the ~~final~~ computation reaches a final state, we will move to a new state q_{blank} which replaces every cell in the input tape with R and then it goes to a new final state q_{final} and gets accepted.

Suppose ~~Let~~ H works in the following way:

> On input $\langle M, w \rangle$, where $\langle M \rangle$ is a TM code and $w \in \{0,1\}^*$, H constructs a new TM M_w as stated above and runs R on $\langle M_w, w \rangle$. If R accepts it accepts, otherwise it rejects.

Thus, H solves the halting problem. But since we know that the halting problem is undecidable, we have a contradiction.

∴ The given problem is undecidable. ∎

② Given as input a triple $\langle M, w, N \rangle$, where $\langle M \rangle$ is a TM code, $w \in \{0,1\}^*$ is its input and N is a positive integer. The problem to decide if M will ever use more than N tape cells in its computation on w is a decidable problem, which we prove below:

Consider a TM M_1 that takes $\langle w, N \rangle$ as input with w, N defined as above. Given a string $w \in \{0,1\}^*$, let w_i denote the ith symbol of w. Then M_1 will output the string w', where w' is ~~dis~~ defined as follows:

$$w_i' = \# \text{ if } i > N$$
$$w_i' = w_i \text{ if } i \leq |w| \text{ and } i \leq N$$
$$w_i' = B \text{ if } i > |w| \text{ and } i \leq N$$

A configuration consists of the state of the central, position of the head and contents of the tape.

Let M_2 be a TM with q states and p tape symbols ~~that most N tape cells)~~ ~~st~~ such that it uses at most N tape cells.

~~Then we get that the total number of available tape cells is~~

∴ The total number of available tape cells is N,

∴ the head can be in a total of N places and p^N possible strings of tape symbols appear, as in each cell, there can be p symbols and M_2 can have q states.

∴ The total number of configurations of M_2 will be qNp^N.

Consider another TM M_3 which works as follows:

On input w', M_3 simulates the TM M.
If the input head tries to read or write on a cell containing a # (i.e., a marked cell), M_3 rejects immediately.
M will be simulated for qNp^N steps or until it halts.
During that time, if the input head never tries to read from a marked cell or write on a marked cell, accept otherwise reject.

Let M_4 be a TM with input $\langle M, w, N \rangle$ which works as follows:
We run M_1 on w to get w' and then run M_3 on w' as input. If M_3 accepts, we accept, otherwise we reject.
Here, M_3 runs for qNp^N steps because if it has not halted in qNp^N steps, it must be repeating a configuration and therefore looping.
In this case, if it has not used any marked cells, the it will never use a marked cell.

Thus, M_4 solves the given problem.

Thus, the problem to decide if M will ever use more than N tape cells in its computation on $w \in \{0, 1\}^*$, is decidable (because M_4 solves it).

(4) A <u>Post Tag System</u> is a finite set P of pairs (α, β) chosen from some finite alphabet and a start string γ.

We say that $a\delta \Rightarrow \delta\beta$ if (α, β) is a pair.

$\overset{*}{\Rightarrow}$ is defined to be the reflexive, transitive closure of $\Rightarrow$, as for grammars.

Let $L_b := \{\langle M, w\rangle \mid M \text{ accepts } w \text{ with a blank tape}\}$.

Now if $\langle M, w\rangle \in L_b$, then the initial ID is qw and final ID is q_f $(q_f \in F)$.

Now we shall make two groups:

<u>Group 1</u>: Consisting of all pairs (a, a) such that $a \in \Gamma$.

<u>Group 2</u>: Consisting of (q_b, cp) if $\delta(q, b) = (p, c, R)$, and (aq_b, pac) if $\delta(q, b) = (p, c, L)$, $a \in \Gamma$.

Since M might have multiple final states, we shall make another group in that case:

<u>Group 3</u>: Consisting of (q_f, ϵ) such that $q_f \in F$

We shall show a general case:

<u>TM</u>: $a_1 a_2 \cdots a_m q b_1 b_2 \cdots b_n \xrightarrow[(p, c, R)]{\delta(q, b)} a_1 a_2 \cdots a_m c p b_2 \cdots b_n$

<u>Post-tag</u>: $a_1 a_2 \cdots a_m q b_1 b_2 \cdots b_n \overset{(a_i, a_i)}{\underset{m-1}{\Longrightarrow}} q b_1 b_2 \cdots b_n a_1 \cdots a_n \overset{(ab_1, cp)}{\Longrightarrow} b_2 \cdots b_n a_1 \cdots a_n cp$

<u>TM</u>: $a_1 a_2 \cdots a_m q b_1 b_2 \cdots b_n \xrightarrow[(p, c, L)]{\delta(q, b_1)} a_1 a_2 \cdots p a_m c b_2 b_3 \cdots b_n$

<u>Post-tag</u>: $a_1 a_2 \cdots a_m q b_1 b_2 \cdots b_n \underset{m-1}{\Longrightarrow} a_m q b_1 b_2 \cdots b_n \overset{(a_m q b_1, p a_m c)}{\Longrightarrow} b_2 b_3 \cdots b_n p a_m c$

If M accepts w with a blank string, then for $q_f \in F$,

$q_0 w \Rightarrow q_f$ and $q_f \xrightarrow{(q_f, \epsilon)} \epsilon$.

If Post tag problem was decidable, the we could create the pairs from M and pass $q_0 w$ as γ and ϵ as δ, and then check iff $\gamma \overset{*}{\Rightarrow} \delta$.

But since L_b is undecidable, we conclude that Post tag problem is undecidable. ∎

(3) We shall show that the set $\mathbb{R}_T$ of all computable real numbers forms a field containing rationals, as follows:

First we show that $\mathbb{Q} \subseteq \mathbb{R}_T$.

We can construct an algorithm ReturnBinary(a,b,n), given any $x = \frac{a}{b} \in \mathbb{Q}$, where $a, b \in \mathbb{Z}, b \neq 0, \gcd(a,b) = 1$, which returns the n^{th} digit of x in its binary representation (using long division). $\therefore x \in \mathbb{R}_T$.

$\therefore \mathbb{Q} \subseteq \mathbb{R}_T$.

Now we shall show that $\mathbb{R}_T$ is a field.

<u>Claim:</u> $(\mathbb{R}_T, +, \times)$ is a ring, where "+" and "×" are the usual addition and multiplication operations.

To prove this, we show the following:

(1) $\mathbb{R}_T$ is closed under addition,

i.e., $a, b \in \mathbb{R}_T \Rightarrow a+b \in \mathbb{R}_T$.

If $a, b \in \mathbb{Q}$, then $a+b \in \mathbb{Q}$ ($\because \mathbb{Q}$ is closed under addition)

$\therefore a+b \in \mathbb{R}_T$

If $a \in \mathbb{R}_T \setminus \mathbb{Q}$ and $b \in \mathbb{R}_T$, let

$$a = p_1 p_2 \cdots p_a \, . \, a_1 a_2 \cdots \boxed{a_n} \cdots a_m \cdots$$
$$b = q_1 q_2 \cdots q_b \, . \, b_1 b_2 \cdots \boxed{b_n} \cdots b_m \cdots$$

where a_n, b_n are the n^{th} pair of digits after the ~~decimal~~ point.

Then, $\forall k > n, (a_k, b_k) \notin \{(0,0), (1,1)\}$.

$\because (a_i, b_i) \in \{(0,1),(1,0)\} \; \forall i > n$, $\therefore a+b \in \mathbb{Q}$.

Thus, their sum will give all 1's ~~after the nth digit~~ from the $(n+1)^{th}$ digit onwards, after the point.

Otherwise, we can find the first $(0,0)$ or $(1,1)$ and add 0 or 1 to the n^{th} digit accordingly.

$\therefore \exists$ a TM that computes $a+b$. A similar approach works for ~~all~~ $a, b \in \mathbb{R}_T \setminus \mathbb{Q}$. For the negative case, the argument is symmetric. ~~everywhere~~

(2) $\mathbb{R}_T$ is closed under multiplication.

Let $a, b \in \mathbb{R}_T$. WLOG, assume $a, b > 0$ since for the negative case, a symmetry argument holds.

~~Let~~

Then, $a \times b = (\lfloor a \rfloor + \{a\}) \times (\lfloor b \rfloor + \{b\})$

$$= \lfloor a \rfloor \lfloor b \rfloor + \lfloor a \rfloor \{b\} + \{a\} \lfloor b \rfloor + \{a\}\{b\}$$

Clearly, $\lfloor a \rfloor \lfloor b \rfloor \in \mathbb{R}_T$ as

$$\lfloor a \rfloor \lfloor b \rfloor = \underbrace{\lfloor a \rfloor + \cdots + \lfloor a \rfloor}_{\lfloor b \rfloor \text{ times}}$$ and $\mathbb{R}_T$ is closed under addition.

Also, $$\lfloor a \rfloor \{b\} = \underbrace{\{b\} + \cdots + \{b\}}_{\lfloor a \rfloor \text{ times}}$$ and $\mathbb{R}_T$ is closed under addition,

$\therefore \lfloor a \rfloor \{b\} \in \mathbb{R}_T$ and similarly, $\{a\} \lfloor b \rfloor \in \mathbb{R}_T$.

We have, $\{a\}\{b\} = \sum_{n=1}^{\infty} \lambda_n$

where, $\lambda_n = \begin{cases} \{a\}\left(\frac{1}{2}\right)^n, & \text{if } b_n = 1 \\ 0, & \text{if } b_n = 0 \end{cases}$

(b_n is the n^{th} digit of b after the point)

~~And since addition is closed over~~

And since $\mathbb{R}_T$ is closed under addition, $\therefore \{a\}\{b\} \in \mathbb{R}_T$.

$\therefore$ $a \times b \in \mathbb{R}_T$, i.e., $\mathbb{R}_T$ is closed under multiplication.

(3) The identities $0, 1 \in \mathbb{Q} \subseteq \mathbb{R}_T$.
Also, given any $a \in \mathbb{R}_T$, the additive inverse $(-a) \in \mathbb{R}_T$.

$\therefore$ $(\mathbb{R}_T, +, \times)$ is a ring.

<u>Claim</u>: $\forall\, a \in \mathbb{R}_T$ (non-zero), the multiplicative inverse $\frac{1}{a} \in \mathbb{R}_T$.

We ~~give~~ have an algorithm for FindInverse(a, n) ~~where~~ ~~call~~ which finds the ~~inverse~~ n^{th} digit of $\frac{1}{a}$, as follows:

- If $a > 1$, then $\frac{1}{a} = 0.a_1'a_2'\cdots$ with first digit as 0. Then do denominator $= \frac{\text{denominator}}{10}$ (i.e., shift ~~one~~ the point one place Left).

- If $a < 1$, then $\frac{1}{a} = 1.a_1'a_2'\cdots$ with first digit as 1. Then do numerator $=$ numerator $-$ denominator and, denominator $= \frac{\text{denominator}}{10}$.

~~We repeat this~~
Repeat this process n times to get the n^{th} digit of a.

$\therefore$ $\mathbb{R}_T$ is a field, as it is a ring and every non-zero element has an inverse.

Yes, $\mathbb{R}_T$ contains irrational numbers, say $\sqrt{3}$ for instance.

Using the usual long division method of calculating square roots, we have:

$$\begin{array}{r|l} & 1\quad 7\quad 3\;\; 2\;\ldots \\ \hline 1 & 3.\,\overline{0\,0}\;\;\overline{0\,0} \\ & 1 \\ \hline 27 & 2\,0\,0 \\ & 1\,8\,9 \\ \hline 343 & \quad 1\,1\,0\,0 \\ & \quad 1\,0\,2\,9 \\ \hline \vdots & \qquad\;\; 7\,1 \end{array}$$

$\therefore \sqrt{3} = 1.732\ldots$. This method uses addition and multiplication operations to calculate the square root. $\therefore \sqrt{3}$ is a computable real number, i.e., $\sqrt{3} \in \mathbb{R}_T$.

Also, $\mathbb{R}_T$ contain transcendental numbers, e for instance

By the ~~Tay~~ Taylor Series expansion, we have

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

This expansion uses addition and multiplication operations (inverses also exist since $(\mathbb{R}_T, +, \times)$ is a field) to calculate the value of e, giving $e = 2.718\ldots$.

$\therefore e \in \mathbb{R}_T$.