Quantum Computing
October 2024, Midsem

SHORT ANSWER TYPE QUESTIONS. EACH QUESTION CARRIES 20 MARKS.

1. (a) Prove that the Eigen values of Hermitian matrices are real.
   (b) Describe how to make classical NAND computation in a reversible way.
   (c) Describe the circuit and the unitary representation of CNOT gate.
   (d) Let $|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$ be a quantum state. Measure the first qubit in $\{|+\rangle, |-\rangle\}$ basis. What is the outcome for the second qubit after the measurement.

2. (a) Draw the quantum circuit corresponding to the SWAP operation. Describe its unitary operation.
   (b) Can you construct an unitary U such that $U(|\psi\rangle|\psi\rangle) = |\psi\rangle|0\rangle$ for any qubit state $|\psi\rangle$?
   (c) Let $Rot_\theta = \begin{pmatrix} cos\ \theta & sin\ \theta \\ -sin\ \theta & cos\ \theta \end{pmatrix}$. Prove that if $Rot_\theta \otimes Rot_\gamma$ is applied to an EPR pair and measured in standard basis, then the probability of getting the same outcome is $cos^2\ (\theta - \gamma)$ and the probability of getting the different outcome is $sin^2\ (\theta - \gamma)$.

3. (a) Let H be the Hadamard operator on one qubit. Show that

$$H^{\otimes n} = \frac{1}{2^{n/2}} \sum_{x,y \in \{0,1\}^n} (-1)^{x \cdot y} |x\rangle\langle y|.$$

   Write an explicit matrix representation of $H^{\otimes 2}$. The trace of a matrix is the sum of the diagonal elements. For any orthonormal basis $\{|i\rangle\}$, prove that trace$(A) = \sum_i \langle i|A|i\rangle$.
   (b) Construct a CNOT gate from a controlled Z gate and two H gate.
   (c) Draw the circuit for Grover's search algorithm and analyze its complexity.
   (d) Sketch a quantum algorithm that will detect a Hamiltonian cycle in a given graph on n vertices if it exits.

4. (a) Construct a quantum circuit that performs the Fourier Transform:

$$|j\rangle \rightarrow \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} \omega^{jk} |k\rangle$$

   where ω is the complex 2^n-th root of unity. Analyze the complexity of the circuit.
   (b) Let

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle.$$

   Show that the operator $2|\psi\rangle\langle\psi| - I$ applied to $\sum_k \alpha_k |k\rangle$ produces

$$\sum_k (-\alpha_k + 2\langle\alpha\rangle) \cdot |k\rangle$$

   where $\langle\alpha\rangle = \sum_k \alpha_k / N$.