From a physiological perspective, the brain must translate the external sources of energy – sight, sound etc – and encode into electrical patterns which the brain can understand. The more elaborately we encode information at the moment of learning, the stronger the memory. The more a learner focuses on the meaning of the information being presented, the more elaborately he/she will process the information. When you are trying to drive a piece of information into your brain's memory systems, make sure exactly what the information means. Do not try to memorize the information by rote and pray the meaning will somehow reveal itself. John Medina

Remember that for positive integers $n \geq 1$, and $0 \leq k \leq n$,

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{1 \cdot 2 \cdot 3 \cdots k} \qquad (\spadesuit)$$

This is same as $\displaystyle\binom{n}{k} = \frac{n!}{k!(n-k)!}$;
convention: $0! = 1 = \binom{0}{0}$ and $\binom{n}{0} = 1$. and $\binom{n}{k} = 0$ if $k > n$..
It is useful to note that ($\spadesuit$) makes sense for any number x (positive or negative real) in place of integer n as long as $k \geq 1$.

1. Consider a set S of $n(\geq 1)$ points. Consider subsets of S which have exactly $k(\geq 0)$ points. How many such subsets are there?

   Answer: $\binom{n}{k}$. Prove it. [must write down!].

2. In what follows $n \geq 1$, and $0 \leq k \leq n$. Show the following.

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

$$\binom{n}{0}\binom{m}{k} + \binom{n}{1}\binom{m}{k-1} + \binom{n}{2}\binom{m}{k-2} + \cdots + \binom{n}{k}\binom{m}{0} = \binom{n+m}{k}$$

$$\binom{n}{k} - \binom{n}{k-1} + \cdots \mp \binom{n}{1} \pm 1 = \binom{n-1}{k}$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

$$\sum_{j=0}^{n} \frac{(2n)!}{[j!]^2[(n-j)!]^2} = \binom{2n}{n}^2$$

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots = n2^{n-1}$$

$$\binom{n}{0}\binom{n}{k} - \binom{n}{1}\binom{n-1}{k-1} + \binom{n}{2}\binom{n-2}{k-2} - \cdots \pm \binom{n}{k}\binom{n-k}{0} = 0$$

Show using induction and binomial theorem that for $n \geq 1$,

$$\binom{n}{1}\frac{1}{1} - \binom{n}{2}\frac{1}{2} + \binom{n}{3}\frac{1}{3} - \cdots + (-1)^{n-1}\binom{n}{n}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Another method: Show the above by integrating (where?) both sides of

$$\sum_{0}^{n-1}(1-t)^k = [1-(1-t)^n]t^{-1}.$$

Show by induction

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^r = 0 \quad \text{or} \quad n!$$

according as $r < n$ or $r = n$.
Another method: Prove it by considering r-th derivative of $(1-e^t)^n$ at $t = 0$.

3. Show that for any positive integer n and any real numbers x, y, z

$$(x+y+z)^n = \sum \frac{n!}{a!b!c!}\, x^a y^b z^c$$

where the sum extends over all non-negative integers a, b, c such that $a + b + c = n$.

In soils of sand, the more you delve the more rush it springs ♢
So too in learning, the deeper you go the more bounty it brings.

Tirukkural

Of the many slips on the slope of life's slippery slips ♢
The worst is the careless word that passes through your lips.

Tirukkural

4. A poker hand means a set of five cards selected at random from usual deck of playing cards.

   (a) Find the probability that it is a Royal Flush - consists of ten, jack, queen, king, ace of one suit. **(b)** Find the probability that it is four of a kind - there are four cards of equal face value. **(c)** Find the probability that it is a full house - consists of one pair and one triple of cards with equal face values. **(d)** Find the probability that it is a straight - consists of five cards in a sequence regardless of suit. **(e)** Find the probability that it consists of three cards of equal face value and two other cards but not a full house. **(f)** Find the probability that it consists of two distinct pairs and another card but does not fall into previous categories. **(g)** Find the probability that it consists of a pair and three other cards but does not fall into previous categories.

5. In how many ways can eight rooks be placed on a chess board so that none can take another and none is on the white diagonal.

6. One mapping is selected at random from the collection of all mappings of $A = \{1, 2, \cdots, n\}$ to itself. What is the probability that the selected map transforms each of the n elements into 1?

   Let $1 \leq i, k \leq n$ be fixed. What is the probability that element i has exactly k pre-images? What is the probability that the element i is mapped to k?

   Suppose distinct elements $i_1, i_2, i_3 \in A$ are given. Also $j_1, j_2, j_3 \in A$ are given. What is the probability that the elements i_1, i_2 and i_3 are transformed to j_1, j_2 and j_3 respectively?

7. One permutation is selected at random from the set of all permutations of $A = \{1, 2, \cdots, n\}$. What is the probability that the identity permutation is chosen?

Assume that distinct points $i_1, \cdots, i_k \in A$ are given and also $j_1, \cdots, j_k \in A$ are given. What is the probability that the selected permutation transforms i_1, i_2 $\cdots$,i_k to j_1, j_2, $\cdots$,j_k respectively?

Let $i \in A$ be given. What is the probability that the permutation keeps i fixed?

What is the probability that the elements 1, 2, and 3 form a cycle in that order? in some order?

What is the probability that all the elements of A form one cycle?

8. Tournament on n vertices is a directed graph, having exactly one arrow (directed edge) between each pair of vertices. Vertices denote players and edge shows who won. A tournament is said to have k-leader property if for every set of k players there is one who beats them all.

   Consider a random tournament with n players. Fix k players. Show: P (no player beats all these k)$= (1-2^{-k})^{n-k}$.

   Show: k-leader tournaments are possible for all large n.

9. (Maxwell-Boltzman expt; n boxes, r balls)

   Show: the probability that box 1 contains exactly k balls is given by $p_k = \binom{r}{k}\left(\frac{1}{n}\right)^k\left(1-\frac{1}{n}\right)^{r-k}$.

   show that the most probable number of balls in box 1 is given by k where $\frac{r+1}{n} - 1 < k \leq \frac{r+1}{n}$; that is
   $p_0 < p_1 < \cdots < p_{k-1} \leq p_k > p_{k+1} > \cdots > p_r$.

   Let m be fixed. As $n \to \infty$ and $r \to \infty$ in such a way that $\frac{r}{n} \to \lambda$ [do you understand this] show that; $p_m \to e^{-\lambda}\frac{\lambda^m}{m!}$.

   Show that the probability of finding exactly m boxes empty equals

   $$p_m(r,n) = \frac{1}{n^r}\binom{n}{m}\sum_{\nu=0}^{n-m}(-1)^\nu\binom{n-m}{\nu}(n-m-\nu)^r.$$

10. (Bose-Einstein expt; n boxes, r balls)

    Show the probability that box 1 contains exactly k balls is given by $q_k = \binom{n+r-k-2}{r-k} / \binom{n+r-1}{r}$

    Let $n > 2$. Show that zero is the most probable number of balls in box 1. That is, $q_0 > q_1 > q_2 > \cdots$.
    As $n \to \infty$ and $r \to \infty$ in such a way that $\frac{r}{n} \to \lambda$ show, for each m; $q_m \to \frac{\lambda^m}{(1+\lambda)^{m+1}}$.

    Show: P(exactly m boxes are empty) $= \binom{n}{m}\binom{r-1}{n-m-1} / \binom{n+r-1}{r}$.

11. Let $A_1, \cdots, A_N$ be events in an experiment. Fix an integer m; $1 \leq m \leq N$. let E be the set of all outcomes ω which belong to exactly m of these events, that is, $|\{1 \leq i \leq N : \omega \in A_i\}| = m$. Show

$$P(E) = S_m - \binom{m+1}{m} S_{m+1} + \binom{m+2}{m} S_{m+2} - \cdots \pm \binom{N}{m} S_N.$$

In matching problem with N cards, the probability of exactly m matches $(0 \leq m \leq n)$ is given by
$p_{[m]} = \frac{1}{m!} \left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots\cdots \pm \frac{1}{(N-m-1)!} \mp \frac{1}{(N-m)!} \right].$

12. Two similar decks of N distinct cards are matched against a similar target deck. Show: probability of exactly m double matches is

$$u_m = \frac{1}{m!} \frac{1}{N!} \sum_{k=0}^{N-m} (-1)^k \frac{(N-m-k)!}{k!}.$$

Show: $u_0 \to 1$ as $N \to \infty$; for each $m \geq 1$, $u_m \to 0$ as $N \to \infty$.

TAs: Mr. Ambaye Om and Mr. Tejas Oke.

Weightage: Midsem 40 and semestral 60.

Books: any book of your choice;
Example, William Feller (vol. 1) or Hoel-Port-Stone (vol. 1) or Sheldon Ross

Most problems are from these and other books.

This is a story about four people named Everybody, Somebody, Anybody and Nobody. There was an important job to be done. Everybody was asked to do it. Everybody was sure that Somebody would do it. Anybody could have done it, but Nobody did it. Somebody got angry about it because it was Everybody's job. Everybody thought Anybody could do it but Nobody realized that Everybody would not do it. It ended up that Everybody blamed Somebody when Nobody did what Anybody could have done. Charles Osgood (?)

13. Find the probability that, in five tosses, a coin falls heads at least three times in succession.
    Find the probability for a head run of at least length five in ten tosses.
    In a sequence of symbols H, T; a run is a maximal sequence of a symbol uninterrupted by the other symbol. For example: $TTTTHHHTHTT$ has 5 runs, two H runs of lengths 3 and 1; three T runs of lengths 4,1,2.

14. Die A has four red faces and two white faces. Die B has two red and four white faces. A fair coin is flipped once; if Heads we use die A forever; if Tails we use die B forever.
    Show probability of red at any throw is $1/2$.
    If first two throws resulted in Red, find the chances of red at third throw? If the first n throws resulted in Red, find the chances that die A is being used?

15. A closet contains n pairs of shoes. $2r$ shoes are chosen at random. What are the chances that there is no complete pair? exactly one complete pair? exactly two complete pairs?

16. Picking card from usual deck with replacement: How many drawings are needed so that chances of obtaining at least one ace is at least $1/2$.

17. A test, developed to detect Covid, gives correct diagnosis for 99% of patients with Covid. It gives wrong diagnosis for 2% of patients not having Covid. In a population one out of 1000 has Covid. A person selected at random from the population tested positive, what are the chances that the person has Covid?

18. Matching N envelopes and letters: What are the chances that there is match at place i given that there is no match at position j.

19. Have two urns each having 20 balls numbered $1, 2, \cdots, 20$. A set of six balls are drawn from each urn. What is the probability that exactly k numbers are common for both samples $(0 \leq k \leq 6)$?
What if the six balls are drawn without replacement from each urn.

20. 10 balls are put in 10 boxes (Maxwell-Boltzman). Given that box number one is empty, what are the chances that only one box is empty? Given that only one box is empty, what are the chances that box one is empty?

21. A fair die is rolled two times. Given that 5 showed at least once what is the probability that it showed both times? exactly once?

22. I have given a letter to Dwiti to post. The chances that she forgets to mail is 0.1. Given that she posted, the chances that the letter reaches the addressee post office is 0.9. Given that it reached that post office, the chances that the postman fails to deliver is 0.1.
Given that the addressee did not get the letter, (i) what is the probability that Dwiti forgot to mail? (ii) Dwiti mailed but the letter did not reach that PO? (iii) it reached the PO but was not delivered?

23. If $P(A) = 1/2$, $P(B) = 1/3$ and $P(A \setminus B) = 1/3$, decide whether A, B are independent or not.

24. I take a sample of size 12 with replacement from $\{1, 2, \cdots, 35\}$ Show that the probability that the first sample item is 23 equals the chances that last sample item is 23.
Is this true for sampling without replacement?

25. A company has three machines A, B, C making bolts. Of the total output, 25% is by A; 35% is by B and 40% by C. It is observed that 3% of A's output; 4% of B's and 5% of C's is defective. One item picked at random was found defective. What are the chance that it was made by A? by B? by. C?

26. Using a machine which produces independent random bits; devise an algorithm to pick a number at random from $\{1, 2, 3, 4\}$;
same to pick at random from $\{1, 2, 3\}$.

27. I make a random 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as follows: the diagonal elements are chosen independently at random from $\{1, 2, 3\}$. The off diagonal elements are selected independently at random from $\{1, \cdots, \max(a, d)\}$ What are the chances that the matrix is singular?

28. Find the chances that a random permutation of $\{1, 2, 3, 4\}$ has exactly two cycles each of length 2? has at least one cycle of length 2? has at least one cycle of length at leat two?

29. Polya scheme of adding c balls: Let $1 \le m < n$ be integers and A, B each stand for any of red/green. Show (time symmetry)
$P(A \text{ at draw } m | B \text{ at draw } n) = P(A \text{ at draw } n | B \text{ at draw } m)$.

30. Let $p_k(n)$ be chances of exactly k green balls in the first n draws in Polya scheme. Show [For ease in writing, interpret $p_{-1}(n) = 0$]

$$p_k(n+1) \quad = \quad p_k(n)\frac{r + (n-k)c}{g + r + nc} + p_{k-1}(n)\frac{g + (k-1)c}{g + r + nc}.$$

$$p_k(n) = \binom{-g/c}{k}\binom{-r/c}{n-k} \Big/ \binom{-(r+g)/c}{n} \qquad 0 \le k \le n.$$

Notation: $\displaystyle\binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{1 \cdot 2 \cdots k}$ for $k \ge 1$ an integer and $\binom{a}{0} = 1$.

31. r alphas and s betas are arranged in a row, at random. Show: the chances that the arrangement contains a total of k runs equals

$$2\binom{r-1}{\nu-1}\binom{s-1}{\nu-1} \Big/ \binom{r+s}{r} \qquad k = 2\nu;$$

$$\left\{\binom{r-1}{\nu}\binom{s-1}{\nu-1} + \binom{r-1}{\nu-1}\binom{s-1}{\nu}\right\} \Big/ \binom{r+s}{r} \qquad k = 2\nu + 1.$$

In a pioneering 2015 study, people were presented with a hypothetical scenario of a self driving car about to run over several pedestrians. Most said that in such a case the car should save the pedestrians even at the price of killing its owner. When they were then asked whether they personally buy a car programmed to sacrifice its owner for the greater good, most said **no**. Yuval Noah Harari.

32. A student is taking a multiple choice exam in which each question has 4 suggested answers with exactly one correct. the student knows the correct answer for sixty percent of the questions. If he knows the answer he chooses the correct answer, otherwise he selects at random.
What is the probability that for a given question he selects the correct answer? Given that he chose the correct answer what are chances that he guessed?

33. Let $A_1, \ldots, A_n$ be events in an experiment. Show that they are independent if the following holds.
(♠) Whenever $B_i = A_i$ or $B_i = A_i^c$ for $1 \leq i \leq n$; then
$P(\bigcap_{1 \leq i \leq n} B_i) = \prod_{1 \leq i \leq n} P(B_i)$.
Conversely show that whenever (♠) holds, then $A_1, \ldots, A_n$ are independent.

34. Let $A_1, \ldots, A_9$ be independent.
(i). Show that the following are independent:
$A_1 \cup A_2$; $A_3 \cap A_4$, $A_5^c \cup A_6^c$, A_7^c, $A_8 \cup A_9^c$.
(ii) show that
$P\{(A_1 \cup A_5) \cap (A_7^c \cup A_8) \,|\, (A_2 \cup A_3^c)\} = P\{(A_1 \cup A_5) \cap (A_7^c \cup A_8)\}$

35. I roll a fair die n times (independently). Let A_i be the event that i-th throw is even; for $1 \leq i \leq n$. Let A_{n+1} be the event that the total sum is even. Show that any n of these $(n+1)$ events are independent. Show that the $(n+1)$ events are not independent.

36. From the set $\{1, 2, \ldots, 100\}$ I take a sample of size 5 with replacement.
(i). Let A_i be the event that i-th item is 99, for $1 \leq i \leq 5$. Are the events independent? Answer the same question for sampling without replacement.
(ii) Let B_i be the event that i-th item is i^2 for $1 \leq i \leq 5$. Answer the same two questions for these events.

37. I have a die, face i appearing in a throw has chance p_i for $1 \leq i \leq 6$, where $p_i > 0$ for all i and $\sum p_i = 1$. I roll it 100 times.
    Let X be the number of throws when even face appears. Find the distribution of X.
    Let Y be the number of times when a face other then 5 appears. Find the distribution of Y.
    Calculate means and variances of X and Y.

38. Throw a die twice independently. Let X and Y be the numbers in the first and second throw. Calculate $E(X \vee Y)$ and $E(X \wedge Y)$. Here $\vee$ is max and $\wedge$ is min.
    Verify. $E(X \vee Y) \ + \ E(X \wedge Y) = E(X) + E(Y)$

39. Calculate the variance of a $P(\lambda)$-variable.
    same question for a $G(p)$-variable.

40. Let $X \sim G(p)$. Calculate $P(X > 30 \mid X > 19)$.

41. Give examples of events A, B where $P(A|B) > P(A)$; where $P(A|B) = P(A)$; where $P(A \mid B) < P(A)$.

42. Suppose $C_1, C_2, \ldots C_n$ is a partition of the sample space. Let A be an event. If $P(A \mid C_i) \leq a$ for all i, then show $P(A) \leq a$. If $P(A \mid C_i) = a$ for all i, then show $P(A) = a$.

43. In this exercise matrices have zero-one entries; arithmetic operations are mod 2.
    Let A, B, C be $n \times n$ matrices and $AB \neq C$. Show that if you select a vector $v \in \{0,1\}^n$ at random (all outcomes are equally likely), then $P(ABv = Cv) \leq 1/2$.

44. Two subsets X, Y of $\{1, 2, \ldots, n\}$ are selected independently and at random from among all subsets. Find $P(X \subset Y)$.
    Find $P(X \cup Y = \{1, 2, \ldots, n\})$

45. We draw cards at random with replacement from usual deck of cards. What is expected number of draws needed till we see all the 52 cards?

46. For a random permutation on $\{1, 2, \ldots, n\}$, what is the expected number of fixed points? What is the expected number of cycles?

47. Consider the random variable X with distribution: $P(X = k) = 1/2^k$ for $k = 1, 2, \ldots$. Calculate $E(2^X)$.

May All Be Happy ♢ May All Be Free From Illness.
May All See The Auspicious ♢ May No One Suffer. Upanishads
If you do not see what is behind that which you see,
you are as blind as blind can be. Tirukkural

48. I throw a fair die. If 4,5 or 6 shows up I put X as that value. If 1,2 or 3 shows up I roll the die again and then put X as the sum of the two faces obtained. calculate the distribution of X.

49. A bag contains balls numbered one to hundred. A sample of size r is taken with replacement. Let X be the maximum of the numbers observed. Find the distribution of X for $r = 1, 2, 3, 4$. Find $E(X)$ for these values of r. What if X were minimum.

50. For any two events A and B, show: $P(A \cap B) \geq P(A) + P(B) - 1$. This is called *a* Bonferroni inequality. The inclusion-exclusion formula we proved is, sometimes, called Poincare formula.

51. What is the most probable value of a $B(n, p)$ variable?
What is the most probable value of a $P(\lambda)$ variable.

52. I have 100 books. I sample with replacement till I obtain seven different books and then stop. Let N be the size of the sample. Find $E(N)$.

53. There are 100 persons from each of 15 different states (total 1500). I take a sample of size 100 with replacement and N is the number of states not represented in the sample. Find $E(N)$.

54. If $X \sim P(\lambda)$, show $E[X(X-1)\cdots(X-r+1)] = \lambda^r$.
if $X \sim G(p)$, show $E[X(X-1)\cdots(X-r+1)] = \dfrac{r!(1-p)^r}{p^r}$.

55. If $X \sim G(p)$, show $P(X \geq i+j | X \geq i) = P(X \geq j)$. and
$E\left(\frac{1}{1+X}\right) = \log\left[p^{p/(p-1)}\right]$.

Participate and Enjoy Tessellate

A University is a seat of learning, not a centre of worship. It believes in the pursuit of knowledge and not in the establishment of a cult. As university men it is our privilege and honour to seek for truth and in this pursuit we should not be deterred by the fear of what we might find.

S Radhakrishnan

56. X, Y are independent rvs on a space.
$X \sim P(\lambda)$ and $Y \sim P(\mu)$, Given $X + Y = n$, find the conditional distribution of X.
$X \sim B(n,p)$ and $Y \sim B(m,p)$, Given $X + Y = k$, find the conditional distribution of Y.
$X \sim G(p)$ and $Y \sim G(p)$, Given$X + Y = n$, find the conditional distribution of Y.

57. Fix $0 < p < 1$. Then $G(n,p)$ is the (undirected) random graph model where we have. n vertices and where edges are chosen independently each with probability p. Let q_n be the probability that the graph is disconnected. Show

$$q_n \le \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}(1-p)^{k(n-k)} \le \sum_{k=1}^{[n/2]}\{n(1-p)^{n/2}\}^k \longrightarrow 0$$

Thus $1 - q_n \longrightarrow 1$. This is expressed by saying that, in this model, almost every graph is connected. A better way to think is that, as n becomes large, our random graph is connected with an overwhelming probability.

58. Let $G = (V, E)$ be a graph (given to you). Choose points of V at random, each with probability $1/2$. Let T be the random set so obtained. Call an edge 'crossing edge' if one end is in T and the other in $V - T$. Since T is random, the number X of crossing edges is a random variable. Show that $E(X) = e/2$ where e is the total number of edges in the graph, that is, cardinality of E.
Deduce that a graph with n vertices and e edges must have a bipartite subgraph with at least $e/2$ edges. A subgraph of (V, E) means a graph (V^*, E^*) where $V^* \subset V$; $E^* \subset E$. Carefully note E^* need not be all of E restricted to $V^* \times V^*$. A graph (V, E) is bipartite if $V = V_1 \cup V_2$ with (i) $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$; (ii) $V_1 \cap V_2 = \emptyset$ and (iii) every edge has one vertex in V_1 and other in V_2.

59. Consider a Erdos-Renyi random graph, where each edge is chosen, independent of others, with probability p. Say that a pair of vertices (unordered) is good, if either they are joined OR they are joined to a common vertex. Let X be the number of bad (not good) pairs. Since the graph is random, X is a random variable. Show that

$$E(X) \leq \binom{n}{2}(1-p^2)^{n-2} \longrightarrow 0$$

Deduce that $P(X \geq 1) \longrightarrow 0$. Conclude that almost every graph has diameter 2. Diameter of a graph is the largest possible distance between pairs of vertices and distance between a pair of vertices is the number of edges in the shortest path joining them — ∞ if no such path.

60. Show that there is a tournament on n players which has at least $n!2^{-(n-1)}$ Hamiltonian paths. A Hamiltonian path is an ordering of the vertices $v_1, v_2, \cdots, v_n$ such that for each $i < n$, $\overrightarrow{v_i v_{i+1}}$ is an edge. Do not confuse with Hamiltonian Cycle.

61. Let $v_1, \cdots, v_n \in R^n$ with $\|v_i\| = 1$ for all i. Show that there exist numbers ϵ_i $(1 \leq i \leq n)$ each ± 1 such that $\|\sum \epsilon_i v_i\| \leq \sqrt{n}$. Also show: there are numbers η_i $(1 \leq i \leq n)$ each ± 1 such that $\|\sum \eta_i v_i\| \geq \sqrt{n}$.

62. Consider placing 3 balls in 3 boxes (Maxwell-Boltzman). Let N be the number of occupied boxes. Let X_i be the number of balls in box i for $i = 1, 2, 3$. Calculate joint distribution of (N, X_3); of (X_1, X_2, X_3).

63. In a sequence of Bernoulli trials, let X be the length of the run started by the first trial. Find its distribution, mean, and variance. Let Y be the length of the second run. Find its distribution, mean and variance. Find joint distribution of X, Y.

64. Suppose that a store buys b (integer) items in anticipation of a random demand Y, non-negative integer valued. Suppose that each item sold brings a profit of Rs π while each item stocked but unsold brings a loss of Rs λ. The problem is to choose b so as to maximize the expected profit. Show that the least integer k satisfying $P(Y \leq k) \geq \pi / (\lambda + \pi)$ is the best choice of b. What if $\pi = \lambda$? What if $\pi = 3\lambda$ or $\lambda = 3\pi$?

Congratulations to all students for Art Work/Tessellate organization.

Worry is a kind of thought and memory evolved to give life direction and protect us from danger. Without its nagging whispers, we would be prone to a reckless Panglossian life style marked by drug abuse, unemployment, bankruptcy, $\cdots$. A modest level of worry is usually best. Too much worry strands us in an agitated state of despair, anxiety and paranoia; too little leaves us without motivation and direction. Worry contributes to life's to-do list, but its relentless prompts are unpleasant and we work to diminish them by crossing items off the list. The bottom line? Stop worrying about worry. It is good for you. Robert Provine

Read all exercises given so far and understand them (not asking to solve all of them) some of you are not even reading them!

65. Let X, Y, Z be independent each $G(p)$. Calculate $P(X = Y)$; $P(X \geq 2Y)$; $P(X + Y \leq Z)$. Let $U = \max\{X, Y\}$ and $V = X - Y$. Show that U, V are independent.

66. Two dice are thrown. let X be the score on the first die and Y be the larger of the two scores. Find joint distribution of X, Y; their means, variances and covariance.

67. In five tosses of a fair coin, let X, Y, Z be, respectively, number of Heads; number of Head runs; length of largest Head run. Calculate their joint distribution. Make bivariate tables for each pair. calculate marginals. Find means variances and covariances.

68. We place n bar magnets end to end in independent random orientation. Like ends attract and unlike ends repel making the system into certain random number of blocks. Find expectation and variance of the number of blocks.

69. $\{X_k\}$ be iid strictly positive discrete rvs with both $E(X)$ and $E(\frac{1}{X})$ finite. Denote $E(X) = a$. Let $S_n = \sum_1^n X_k$.

    Show $E(\frac{1}{S_n})$ exists (yes, need to show) and $E(\frac{X_k}{S_n}) = \frac{1}{n}$ for $1 \leq k \leq n$.

    Show $E(\frac{S_m}{S_n}) = \frac{m}{n}$ if $m \leq n$.

    Show $E(\frac{S_m}{S_n}) = 1 + (m - n)aE(\frac{1}{S_n})$ if $m \geq n$.

70. let S_n be the number of matches with n cards (or letters-envelopes). We saw $E(S_n) = 1$ no matter what n is. Show $Var(S_n) = 1$.

71. X non-negative integer valued, show $E(X) = \sum_1^\infty P(X \geq n)$. Justify your steps (we are slowly entering analysis!)

72. Let X be any random variable (discrete). Define $F(a) = P(X \leq a)$ for $a \in R$. This function is called the Distribution Function (DF) of X.

    (i) Show F is monotone non-decreasing; $0 \leq F(a) \leq 1$.

    $\lim_{a \to -\infty} F(a) = 0$ and $\lim_{a \to \infty} F(a) = 1$.

    (ii) Show that F is right continuous.

    (iii) Show that F has left limit at every $a \in R$. That is, given $a \in R$; there is a number α such that whenever you take a sequence of numbers $x_n \uparrow a$; $x_n < a$ for every n, then $F(x_n) \to \alpha$. This number α is denoted $F(a-)$.

    (iv) Show that for every a; $P\{X = a\} = F(a) - F(a-)$.

    Deduce that $P\{a\} = 0$ iff F is continuous at a, equivalently, F is left continuous at a.

73. Here is a nice DF. Enumerate all rationals: $\{r_n : n \geq 1\}$. Let X be a random variable which takes the value r_n with probability $1/2^n$ for $n \geq 1$. Let F be its DF. Show that F is continuous exactly at irrational numbers: (i) by using earlier results (ii) directly verifying definition of continuity.

    Thus there is a function on R which is continuous exactly at irrationals. Think: Is there a function on R which is continuous exactly at rationals?

74. Suppose X is a random variable having density $f(x)$. Consider the random variable $Y = 10X$. Find the density of Y. Suppose $Z = X+39$. Find the density of Z.

    Suppose X is standard normal. Let $Y = 33X + 29$. Find the density of Y.

75. Suppose $X \sim exp(\lambda)$. Show that $P(X > a+b \mid X > a) = P(X > b)$ for $a > 0$ and $b > 0$.

If all insects disppear from this planet ♦
all other forms of life will PERISH.
If all humans disappear from the planet ♦
all other forms of life will FLOURISH.

76. Let $X \sim \exp(1)$ and $c > 0$. Let $Y = X^{1/c}$. Show that Y has density

$$g(y) = cy^{c-1}e^{-y^c} \qquad y > 0$$

This is called Weibull distribution and is also used in modelling.
Fix $\theta > 0$ Show that the following function is a density function.

$$f(x) = \frac{\theta}{x^2} \qquad x > \theta.$$

This is called Pareto density and is used for modelling in Economics.
Let $X \sim \Gamma(\lambda)$. Find the density of $Y = 1/X$.
Let $X \sim Unif(0,1)$. Put $Y = \sin(\frac{1}{2}\pi X)$. Find density of Y.
Do the same if $Z = \sin(\pi X)$. What if $Z = \sin(2\pi X)$?
Let $X \sim N(0,1)$. Find the density of $Y = |X|$ (Think).
Let $X \sim N(0,1)$. Find the density of X^2.

77. Let (X, Y) have joint density $f(x, y)$. Show that X has density given by $g(x) = \int_{-\infty}^{\infty} f(x, y)dy$.
X and Y are random variables on a space with densities $f(x)$ and $g(y)$ respectively. We say X, Y are independent if their joint density is given by $f(x, y) = g(x)h(y)$.
If X, Y are independent then show that for any two intervals $I, J \subset R$ we have $P(X \in I, Y \in J) = P(X \in I) \times P(Y \in J)$.

78. (X, Y) have joint density $f(x, y) = 21x^2y^3$ for $0 < x < y < 1$. Find densities of X and Y. Find

$$P(1/2 < X < 3/4; 1/4 < Y < 1); \quad P(XY < 1/2); \quad P\{\frac{X}{Y} > 1/4\}$$

79. (X, Y) have joint density $f(x, y) = x + y$ for $0 < x < 1$ and $0 < y < 1$. Find densities of X and Y. Find

$$P(X < Y); \qquad P(XY < 1/4); \qquad P(X^2 + Y^2 > 1/4)$$

80. A company guarantees machines to be trouble free for the first five weeks after purchase. It is known that the time to first failure is exponential with mean 10 weeks. I bought one such machine. What is the probability that it violates the guarantee?
One such machine is placed in service in the hostel. Upon inspection exactly 8 weeks later it is found to have failed at some time earlier. Given this, what is the probability that it actually failed in the first five weeks?

81. Pick a point at random from the unit disc. Find the density of the distance of that point from the origin. That is, (X, Y) has joint density $f(x, y) = 1/\pi$ for (x, y) in the unit disc and zero otherwise. Find the density of $\sqrt{X^2 + Y^2}$

82. I pick a point X at random from $(-\pi/2, +\pi/2)$. Find the density of $Y = \tan X$.

83. X, Y are independent exp(1). Find density of $Z = X + Y$.

    Do the same if X, Y are independent uniform $(0, 1)$. Sketch the curve.

I am worried about stupid, it is all around us. Congress debates an issue, both sides appear wrong. Representatives can not seem to make a reasonable argument. In customer service, the person you are talking to is reading from a script, can not deviate because they do not know what they are talking about. News has become a mouth piece for views that can be parroted by their listeners. Challenging beliefs is not part of the news anymore. I worry we stopped asking 'why'. We say prescription drug works miracles, but we fail to ask what we really know about what else it does. We glorify stupid on TV shows, showing what dumb things people do, so that we can all laugh at them. I worry behind this glorification of stupidity and the refusal to think hard about real issues are big corporations who make a great deal of money on this. I am worried that people can not think, can not reason from evidence, and do not even know what would constitute evidence. People do not know how to ask the right questions. $\cdots\cdots$.

Roger Schank

Relax. You can all do these. Play, be patient, think, think and think.

84. A subset A of integers is said to be sum-free, if sum of two things in A is not in A. More precisely, if you **can not** find $a_1, a_2, a_3 \in A$ such that $a_1 + a_2 = a_3$. We did not say that these a_i be distinct. If $0 \in A$, A is not sum free. if $a, 2a \in A$ then A is not sum-free.
Given a set B of non-zero integers, we can find a sum-free subset $A \subset B$ having at least one-third of the number of elements of B. Here is how (See the power of probability!).
Say, $B = \{b_1, \ldots, b_n\}$. Pick a prime $p > 2 \max\limits_{1 \leq i \leq n} |b_i|$ which is of the form $p = 3k + 2$. Show that such a p exists.
Show $C = \{k+1, k+2, \ldots, 2k+1\}$ is sum-free in $Z_p = \{0, 1, \ldots, p-1\}$ where addition is done modulo p.
Pick x at random from the set of non-zero elements of Z_p and set $d_i = x \cdot b_i \ mod \ (p)$. Show that, for fixed i, as x ranges over the non-zero elements of Z_p, so does d_i. Conclude that $P(d_i \in C) \geq 1/3$.
Show that the expected number of b_i such that $d_i \in C$ is at least $n/3$.
Conclude that there is x, $1 \leq x \leq p-1$ and an $A \subset B$ with $|A| \geq n/3$ such that $x \cdot a \in C$ for all $a \in A$. Here $x \cdot a$ is taken mod (p).
Show that A is sum-free.

85. A bakery makes 80 loaves of bread daily of which ten are underweight. An inspector weighs five loaves at random. What is the probability that an underweight loaf will be discovered? Simplify using calculator. (We shall NOT have such problems, needing calculator, in exam.)

86. I have a coin whose chance of heads is p. I toss it $n+1$ times. For $1 \le i \le n$, let X_i be one if both i-th and $(i+1)$-th tosses yield heads; and zero otherwise. find the mean and variance of the sum $\sum X_i$.

87. Consider a die with k faces, face i turning up on a throw is p_i for $1 \le i \le k$ where $\sum p_i = 1$. Roll the die n times independently. Let X_i be the number of times face i appears.
Find the joint distribution of $(X_1, X_2, \cdots, X_k)$. This is called multinomial distribution: $M(n; p_1, \cdots, p_k)$. If you denote $X = (X_1, X_2, \cdots, X_k)$, one says $X \sim M(n, p_1, \cdots, p_k)$.
What is the marginal distribution of X_5.
What is the distribution of (X_1, X_2, X_3). What is the distribution of (X_1, X_3, X_5, X_7)
What is the distribution of $X_1 + X_3 + X_5$.

88. Suppose that the life length of a radio tube can be modelled as continuous random variable X with density $f(x) = 100/x^2 \quad \text{for} \quad x > 100$. What is the conditional probability that a tube will last more than 200 hours if it is still functioning after 150 hours of service.
What is the probability that if three such tubes are installed in a set, exactly one will have to be replaced after 150 hours of service.
What is the maximum number of tubes that may be inserted into a set so that there is a probability of at least 0.25 that after 150 hours of service all of them are still functioning?

89. The speed of a molecule in a uniform gas at equilibrium is a random variable V with pdf given by

$$f(v) = a\, v^2\, e^{-b\, v^2} \quad \text{for} \quad v > 0$$

where $b = m/2kT$. Here k is Boltzman constant, m is the mass of the molecule and T is the absolute temperature. Evaluate the constant a in terms of b.
Derive the distribution of the kinetic energy $W = mV^2/2$.

90. Let X, Y be independent uniform$(0,1)$. Find the joint density of U, V where $U = X + Y$ and $V = X - Y$. Find density of $X + Y$. Find density of $X - Y$. Sketch their graphs.
Find density of $Z = XY$. Find density of $W = X^{85}$.

91. Consider an equilateral triangle with sides having length s. A point is selected at random from a side. Let X be its distance from the opposite vertex. Find the density of X. (Take time to understand, no hurry).

Our greatest weakness lies in giving up ♦ The most certain way to succeed is always to try just one more time. Thomas Edison

It is only natural that if one has already expended considerable effort on a problem, then one is reluctant to leave it unsolved. In some sense it becomes a personal struggle: the problem against yourself. And if the ultimate goal is to complete one's understanding, then one is very reluctant to accept defeat. And this feeling of personal antagonism towards a problem is a natural tendency. At least it has remained with me all my life. S Chandrasekhar

92. Random Walk is an **I**dea.
I toss a coin whose chance of heads in a single toss is p, where $0 < p < 1$. Heads up, I move one step forward; tails up I move one step backward (on the space described below) and repeat. Write down the transition probabilities in the following cases. In each case calculate distribution of X_2 after choosing some initial starting point X_0.
(a) The state space is all integers. This is usual RW or *unrestricted* RW. We did in class with $p = 1/2$.
(b) The state space is $\{0, 1, 2 \cdots\cdots\}$. If you are at 0 then move to 1. This is RW *reflected* at zero.
(c) The state space is $\{0, 1, 2 \cdots, 99\}$. If at 0 or at 99, do not move; stay there forever. This is RW with two *absorbing barriers*.
(d) State space is as in (c). If at 0, move to 1. If at 99, move to 98. This is RW with two *reflecting barriers*.
(e) State space is as in (c). Fix two numbers $0 < r_i < 1$ for $i = 0$ and $i = 99$. If at 0 stay there with probability r_0 and go to 1 with probability $1 - r_0$. Similarly do at 99 using r_{99}. This is RW with two *elastic* barriers.
(f) State space is as in (c). If at 0, move to 1. If at 99, stay there. This is RW with one absorbing barrier and one reflecting barrier.
(g) State space is as in (c). Treat 0 as state next to 99 and 99 as state preceding 0. Equivalently, think of the states arranged in a circular/cyclic order. There is no need to specify anything more for the motion to continue. This is *cyclic* RW.
(h) You can think of analogues in higher dimensions, but do not spend time.

93. State space consists of the 64 squares of a chess board. There is only one piece on the board, say, a knight on one square. From a state, you select one of its possible Knight moves at random and move there. Is this irreducible chain? What is the period? What is the mean time taken by the knight to return to the starting position? Do the same thing with another chess piece of your choice.

94. State space consists of the 52! arrangements of the usual deck of playing cards. Start with any one arrangement (think of vertical stack). Here are some laws of motion. In each case explain the transition matrix:
(A) interchange bottom card with a randomly selected card.
(B) Select a card at random and put it on top.
(C) select a card at random, interchange its neighbours(think cyclic, top card and bottom card are neighbours).
(D) Pick two cards (withe replacement) at random and interchange them. What if it is without replacement.
(E) Cut at random and interchange the stacks. That is, pick a number at random from among $\{1, 2, \cdots, 51\}$. The stack is now split into two parts: initially: top $[1, k]$ and bottom $[k+1, 52]$. New stack: new 1=old $(k+1)$; new 2 = old $(k+2)$; new $(52-k)$ = old 52; and new $(52-k+1)$ = old 1, etc new 52 = old k.

95. A die is consecutively turned from one face to any of the four neighbouring faces with equal probability and independent of the preceding turns. Find $\lim_n p_{66}^{(n)}$.

96. (Gambler's Ruin) consider random walk with two absorbing barriers $0, N$. state space is $\{0, 1, 2 \cdots N\}$ and the transition mechanism is: win a rupee with probability p or loose a rupee with probability $1 - p$; From i go to $i + 1$ or $i - 1$ with these probabilities (?).

Let u_i = probability of getting absorbed at N starting from i.
Argue: $u_0 = 0, u_N = 1$. Find a recurrence relation for $\{u_i\}$. Show
$u_i = \dfrac{1 - [q/p]^i}{1 - [q/p]^N}$ when $p \neq 1/2$; $\quad u_i = i/N$ if $p = 1/2$.
Here $q = 1 - p$.

97. $X_{n\times 1} \sim N_n(\mu, \Sigma)$ and A is $m \times n$ nonsingular matrix. Find the distribution of $Y = AX$.

98. Let $X_1, \ldots, X_n$ be i.i.d. $\exp(\lambda)$ variables. Find the density of $\sum X_i$.

Farady effect (by Arnold): Lectures which really teach will never be popular; lectures which are popular will never teach.
Neils Bohr: Clearness and truth are in quantum complementarity relation.
V A Rokhlin: Humanity is moving towards bureaucratization where an all-powerful bureaucratic apparatus will be suppressing everything alive and creative that still exists.

99. Intuitively feel and then rigorously show the following for a finite state chain with transition matrix P:

$$P(X_{35} = k; X_{29} = j \,|\, X_1 = i_1; X_{11} = i_{11}; X_{13} = i) \ = \ p_{ij}^{(16)} p_{jk}^{(6)}$$

100. Rigorously argue that the Bose-Einstein Chain does pick a B-E configuration at random.

101. Suppose that you have a finite state chain (X_n) starting with stationary distribution π and π obeys $\pi(i)p_{ij} = \pi(j)p_{ji}$ for all i, j. Then it appears same whether you look forward or backward. This means the following:

$$P(X_5 = i, X_6 = j, X_7 = k; X_8 = l) = P(X_8 = i; X_7 = j, X_6 = k, X_5 = l)$$

In other words, the 'path' $\widetilde{ijkl}$ has same probability as the path $\widetilde{lkji}$. Such chains are called reversible.

102. Consider usual one dimensional random walk. Show that $p_{ij}^{(n)} \to 0$ for each i and j.

103. We showed for any $a, b > 0$ the following integral is finite.

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx$$

Show that for any $a, b, c > 0$ the following integral is finite and find its value.

$$\iint\limits_{0<x,y,x+y<1} x^{a-1}y^{b-1}(1-x-y)^{c-1}dxdy$$

(Discipline: Do not try hard to get lost, recall how you are supposed to integrate in our course and DO so.)

Show that for any $a, b, c, d > 0$ the following integral is finite and find its value.

$$\iint\limits_{0<x,y,z,x+y+z<1} x^{a-1}y^{b-1}z^{c-1}(1-x-y-z)^{d-1}dxdydz$$

Use induction and show that the following integral is finite and calculate its value: $a_1, a_2, \ldots, a_{n+1} > 0$

$$\int\cdots\int\limits_{0<x_1,x_2,\ldots,x_n,\sum x_i<1} x_1^{a_1-1}x_2^{a_2-1}\ldots x_n^{a_n-1}(1-\sum x_i)^{a_{n+1}-1}dx_1\ldots dx_n$$

In particular if $c = c(a_1, a_2, \ldots, a_{n+1})$ is the value of this integral, then the following is a density function:

$$f(x_1, \ldots, x_n) = \frac{1}{c}\, x_1^{a_1-1}x_2^{a_2-1}\ldots x_n^{a_n-1}(1-\sum x_i)^{a_{n+1}-1}$$

in the region

$$\Delta = \{(x_1, \ldots, x_n) : x_1, x_2, \ldots, x_n > 0;\ \sum x_i < 1\}$$

(For points not in this region, density is zero).
This is called Dirichlet density and $(X_1, X_2, \ldots, X_n)$ having this density is said to have Dirichlet distribution. This is useful in Bayesian inference.

104. Mayur, Ryan, Vedant enter a post office with two counters. Service times for the three are independent $\exp(\lambda)$ variables. M and R start their service at the two counters immediately. As soon as one service is finished, immediately service of V will start at that counter. What are the chances that V is the last to leave the post office?

Furious with the humiliating 142 rank in the World Press Freedom Index, the Union cabinet secretary set up a monitoring committee. There were 11 bureaucrats and Government-controlled-institution researchers in a committee of 13 and just 2 journalists. The draft report reflected nothing of the serious issues raised in the meetings. So I submitted an independent or dissenting note for inclusion in it. At once, the report, the committee, everything – **vanished**. RTI enquiries failed to unearth the report – on freedom of press. The original exercise was not even investigative journalism – it was **investigating** journalism, as functioned in India. And it disappeared at the drop of a dissent note. P. Sainath

105. I have the set $S = \{1, 2, \cdots, 100\}$. I want to pick a subset of S at random. Remember there are 2^{100} subsets.
Someone told me the following. You just toss a fair coin 100 times. Let A consist of all i such that i-th toss resulted in heads. Then A is a 'random set'. Show this.

106. As earlier $S = \{1, 2, \cdots, 100\}$. Among all subsets of S with cardinality 50, I want to pick one at random. Remember there are $\binom{100}{50}$ such subsets.
Some one told me the following. Pick a permutation π of S at random. Then independently pick $1 \leq i \leq 100$ at random. Then the following set is such a set: $A = \{\pi(i), \pi(i+1), \cdots, \pi(i+49)\}$. Show this.

107. As earlier $S = \{1, 2, \cdots, 100\}$. I want to pick a permutation at random. Instead of writing 100! permutations and doing this, someone said I can do the following. Do sampling without replacement (size 100) from S. The ordered list so obtained is a random permutation. Show this.
In this second experiment, you are selecting 100 times; but each time from a small manageable set.

108. I want to pick a number at random from $(0, 1)$. This is a huge task. I can only approximate.
I fix an error bound, say, 0.0001. If X is the number picked then, instead of demanding that $P(X \in I)$ should equal length of I; we only demand that the difference between these two must be at most 0.0001 *whatever* be the interval.
Someone told me the following. Fix a large number k. You toss a fair coin k times and think of the result as dyadic expansion of a number

(Heads = 1, Tails = 0). This will do. Show this. How large should I fix k. What if the error bound is 2^{-30}.

109. Consider a finite group G and a probability vector $\{p(g) : g \in G\}$. Here is random walk driven by p on G: if at g, select h according to p and move to hg. What is the transition matrix?
Show: The chain is irreducible iff $\{g : p(g) > 0\}$ generates G.
Show: uniform probability on G is invariant.

110. I have a random variable with moments μ_k, that is, $\mu_k = E(X^k)$ for $k \geq 0$, where, of course, $\mu_0 = 1$. Show that the following matrix is non-negative definite.

$$\begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_k \\ \mu_1 & \mu_2 & \cdots & \mu_{k+1} \\ \mu_2 & \mu_3 & \cdots & \mu_{k+2} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mu_k & \mu_{k+1} & \cdots & \mu_{2k} \end{pmatrix}.$$

111. Let $v_1, \ldots, v_m$ be vectors in R^n. Show that the matrix

$$A = ((a_{ij})) \qquad a_{ij} = \langle v_i, v_j \rangle \qquad i \leq i, j \leq m$$

is non-negative definite. Show that it is positive definite iff the vectors are independent.

112. Suppose that X is a nonnegative random variable. Fix a number $\delta > 0$.
Define the following random variable X_δ:
X_δ takes the value $k\delta$ iff $k\delta \leq X < (k+1)\delta$; for $k = 0, 1, 2 \cdots$.
This is called " discretization" in mathematics, "grouping" in statistics and "quantization" in physics.
Assume that X has density f which is bounded continuous on a bounded interval and zero outside the interval. Show rigorously that $E(X_\delta)$ converges to $\int x\, f(x)\, dx = E(X)$ as $\delta \to 0$.

113. Suppose that a random variable X has density f. Median of X is any number c such that

$$P(X \leq c) = \frac{1}{2} = P(X \geq c); \quad (i.e.) \quad \int_{-\infty}^{c} f(x)dx = \frac{1}{2}$$

Show that a median always exists.
Assume that X has a unique median m. For any real number b, show (assume needed integrals exist)

$$E(|X - b|) \quad = \quad E(|X - m|) + 2\int_m^b (b-x)\, f(x)\, dx$$

For what b is $E(|X - b|)$ minimum. What if median is not unique?

114. $X \sim Poisson(n)$. If n is very large explain why we can approximate the probability $P(n + a\sqrt{n} < X < n + b\sqrt{n})$ by the number $\int\limits_a^b \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$. Here $a < b$.
Show that as $n \to \infty$,

$$\sum_{k \geq n} \frac{n^k}{k!} e^{-n} \to \frac{1}{2}.$$

Akbar's successful reign and patronage led to wonderful flowering of Indian intellect. Akbar's mantle as an eclectic and peacemaker in religion fell on his great grandson Dara Sukoh who openly declared he had found the fullest pantheism (Tawhid) in the Vedanta only. Prepared Persian translations of 50 Upanishads: Majma-'Ul-Bahrain; or mingling of the two oceans. Paper was introduced by Muslims. The illumination of manuscripts is an art which we owe to the Mughal Empire. We owe to Muslims the practice of diffusing knowledge by copying and circulation of books. Hindu medicine had been arrested long ago, Muslim medical science was daily progressing by keeping in touch with the west . . . . Sir Jadunath Sircar in Sir William Meyer Lectures, Madras University, 1928
(National Council for Educational Ruination and Tragedy differs.)

115. Given $-\infty < a < b < \infty$; Given a real continuous function f on $[a, b]$; Given $\epsilon > 0$; exhibit a polynomial P such that
$\sup\{|f(x) - P(x)| : a \leq x \leq b\} < \epsilon$.

Deduce the following: Given a continuous $f : R \to R$; there is a sequence of polynomials $\{P_n\}$ such that
$P_n \to f$ uniformly *over every bounded interval.* This means whatever be real numbers $a < b$, $\lim_n \ [\ \sup_{a \leq x \leq b} |f(x) - P_n(x)|\ \ ] = 0$.

116. Define
$F(x) = 0$ for $x < 21$
$F(x) = 0.2$ for $21 \leq x \leq 23$
$F(x) = 0.2 + \frac{x-23}{4}$ for $23 \leq x \leq 25$
$F(x) = 0.7$ for $25 \leq x < 345$
$F(x) = 1$ for $x \geq 345$
If $X \sim F$, Find $P(21 < X < 22)$ $P(21 \leq X \leq 22)$
$P(22 \leq X \leq 24)$ $P(24 \leq X \leq 345)$.

117. I toss a coin whose chance of heads is 0.3. If heads up I pick a number at random from the interval (0, 3). If tails up I pick a number at random from the interval (2, 4). Let X be the number. Find distribution function of X. Does it have a density?

118. I toss a coin whose chance of heads is 0.3. If heads up, I pick a number at random from $[0, 4\pi]$. If tails up, I pick a number at random from among the set of points in the same interval where the sine function is

zero. Let X be the number. Find distribution function of X. Does it have a density?

119. I pick a point at random from $(0,1)$, say X. I put $Y = X$. Do you think (X,Y) has joint density $f(x,y)$?

120. Let X be uniform $(0,1)$. Find a function of X, say, $Y = f(X)$ which is a Poisson variable with parameter 29. Find a function of X, say, $Z = g(X)$ which is exponential variable with parameter 29.

121. Complete the Details of Dice interpretation of BE statistics.
Consider $n(> 1)$ balls into r boxes as follows: Pick an r-faced die at random. This means pick a point at random from the set

$$\Delta = \{(p_1, \ldots, p_{r-1}) : 0 < p_1, \ldots, p_{r-1}, \sum p_i < 1\}$$

Remember picking a point at random from a set means having density equal to $1/v$ on the set and zero outside. v is volume of the set.
Roll this die once for each ball and place the ball as dictated by the throw. Given $(k_1, \ldots, k_r)$ find probability of the event: (k_i balls in box i for each i). You must calculate v and the integrals that appear.

122. (Not for exam) Alphabet: $1, 2, 3, 4$. Here are some binary codes.

The code $\{0; 010; 01; 10\}$ is not uniquely decipherable code. Argue

The code $\{10; 00; 11; 110\}$ is uniquely decipherable. Show.

The code $\{0; 10; 110; 111\}$ is prefix free code; that is, no code is the beginning segment of another code. Such codes are also called instantaneous codes. Show such codes are uniquely decipherable.

123. (Not for exam) Alphabet: $\{1, 2, 3, 4\}$
with probabilities $P = \{1/2; 1/4; 1/8; 1/8\}$.

Consider the code $\{0; 10; 110; 111\}$. Calculate entropy $H(P)$ and expected code length.

124. (Not for exam) Alphabet: $\{1; 2; 3\}$ with uniform probability P and code $\{0; 10; 11\}$. Calculate $H(P)$ and expected code length.

125. (Not for Exam)

Here is frequencies (in percentage) of letters in English language Copied from:

https://www3.nd.edu/ busiforc/handouts/cryptography/letterfrequencies.html
I have not checked if they add to 100. please do.

E 11.1607 A 8.4966 R 7.5809 I 7.5448 O 7.1635 T 6.9509 N 6.6544.

S 5.7351 L 5.4893 C 4.5388 U 3.6308 D 3.3844 P 3.1671 M 3.0129

H 3.0034 G 2.4705 B 2.0720 F 1.8121 Y 1.7779 W 1.2899

K 1.1016 V 1.0074 X 0.2902 Z 0.2722 J 0.1965 Q 0.1962

See the Morse code in Wikipedia, consider it as binary code: ($\bullet = 0$) and ($- = 1$) — even though in practice there are norms for lengths of these dot and dash, we shall take code length as just number of binary digits in the code sequence. We consider only letter alphabet. Calculate entropy, Calculate average code length.

$\diamondsuit \spadesuit \clubsuit \heartsuit$

Exam on April 26 for two and half hours 9:30 am to 12 noon in the Hall as announced.

We shall have Five questions each two parts.

If something is turning out to be lengthy/ugly then surely you are on wrong track.

Syllabus: all we have done as mentioned in the notes.

Justification of steps is important. You can use, by quoting, results proved in class/notes; but anything from HA – if used – must be worked out.

Discuss with TAs before next Thursday if you have any doubts. No discussions with me during exam week. That week is for YOU to revise.

Good Luck.