Discrete Mathematics Assignment 4

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Problem 1:
Solve the following two questions on trees

- Let G be a tree, which we consider as the network of roads in a medieval country, with castles as nodes. The King lives at node r. On a certain day, the lord of each castle sets out to visit the King. Argue carefully that soon after they leave their castles, there will be exactly one lord on each edge.

- If we delete a node v from a tree (together with all edges that end there), we get a graph whose connected components are trees. We call these connected components the branches at node v. Prove that every tree has a node such that every branch at this node contains at most half the nodes of the tree.

Solution 1:

- Suppose that initially, all lords are at their respective castles and the King is at node r. When each lord starts his journey towards the King, he must take an edge that leads to a neighbouring node. If two lords start their journey along the same edge, then they will eventually meet at some point along that edge. At that point, one of the lords will turn around and head back to his castle, while the other will continue on to the King. Thus, at most one lord can travel along each edge at any given time.

  Now suppose, for the sake of contradiction, that there exists an edge e with no lord traveling along it. Let u and v be the nodes on either end of e. Since G is a tree, there is a unique path from u to v. Let w be the last node on this path that is not on e. Since e has no lord on it, all lords that started at nodes in the subtree rooted at v must have passed through w at some point. This is because, if they didn't, they would have had to cross e to reach the King, which we assumed is impossible. Thus, there is at least one lord in the subtree rooted at v that has passed through w. However, there must also be at least one lord in the subtree rooted at u that has passed through w. Otherwise, all lords from u would have had to pass through v to reach the King, which is impossible since e has no lord on it. Therefore, there are at least two lords at w, which is a contradiction.

  Therefore, we conclude that after each lord sets out from his castle, there will be exactly one lord on each edge.

- We will prove the given statement by contradiction. Assume that there is no node in the tree such that every branch at this node contains at most half the nodes of the tree. Then for any node v in the tree, there must be a branch at v that contains more than half of all nodes in the tree.

  Let us start at an arbitrary node v in the tree. Suppose the branch B at v contains more than half of all nodes in the tree. Then we move along the edge leading to B and consider the tree rooted at B. Since B contains more than half of all nodes, the remaining branches contain at most half of all nodes. Thus, there must be a branch B' in the remaining branches at v that contains more than half of all nodes in the subtree rooted at B'. We repeat this process, always moving to a branch that contains more than half of all nodes.

  Since the tree is finite, this process must eventually stop. We cannot backtrack to a previously visited node, because doing so would mean that there is an edge whose deletion results in two connected components, both containing more than half of the nodes, which contradicts the assumption that every branch at each node contains more than half of all nodes. Therefore, we must get stuck at a node u such that each branch at u contains at most half of all nodes.

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Problem 2:
Prove that in a connected graph G with at least three vertices, any two longest paths have a vertex in common.
Solution 2:
Suppose $P_1 = (u_1, u_2, \ldots, u_n)$, $P_2 = (v_1, v_2, \ldots, v_n)$ be two longest paths in G of length n with no vertices in common. Since G is connected, so there is a path P connecting P_1 and P_2. Therefore, there exists some last vertex u_i of P on P_1 and some first vertex v_j of P on P_2 for some $1 \leq i, j \leq n$. Hence, the path P' connecting u_i to v_j shares no common vertex with $P_1 \cup P_2$ other than u_i and v_j. Let $P' = (u_i, w_1, w_2, \ldots, w_k, v_j)$, $k \geq 1$, be the path. Without loss of generality, we can assume $i, j \geq \frac{n}{2}$. Then we can construct a new path $P^* = (u_1, u_2, \ldots, u_i, w_1, w_2, \ldots, w_k, v_j, v_{j-1}, \ldots, v_1)$, which has length $\geq i+j+1 \geq n+1$, which is a contradiction to the assumption that the longest paths of G has length n. ■

Problem 3:
(Graphic matroid) Let (V, I) and (V, J) be two forests on the same vertex set V where $|I| < |J|$. Show that there is an edge $j \in J$ such that $(V, I \cup \{j\})$ is also a forest.
Solution 3:
Let $M = (V, E)$ be the graphic matroid associated with the undirected graph $G = (V, E)$, and let I and J be two forests of G. Since I is a forest, it does not contain any cycles. Thus, I induces a forest matroid $M_I = (V, E_I)$, where E_I is the set of edges in I. Similarly, J induces a forest matroid $M_J = (V, E_J)$. We know that $|I| < |J|$, which means that M_I has fewer edges than M_J. Therefore, there must be at least one edge $j \in E_J \setminus E_I$ that can be added to I without creating a cycle. This is because if we were to add an edge $j \in E_J \setminus E_I$ to I, creating a cycle C, we could remove an edge $i \in E_I \cap C$ to obtain a forest with $|I \cup \{j\}| \geq |I|+1$ edges, contradicting the fact that I is a forest with $|I|$ edges. Thus, j can be added to I to obtain a new forest $I \cup \{j\}$. Moreover, since $j \notin E_I$, adding j to I cannot create a cycle, which implies that $(V, I \cup \{j\})$ is also a forest. Therefore, there exists an edge $j \in J$ such that $(V, I \cup \{j\})$ is a forest. ■

Problem 4:
Let K_n be the complete graph on the vertices $[n]$ and $\mathbb{F}$ be a field. Let $x_1, x_2, \ldots, x_n$ be the standard basis for $\mathbb{F}^n$. For each $1 \leq i < j \leq n$, we associate a vector in $\mathbb{F}^n$ to the edge $e = \{i, j\}$ of K_n given by $x_e = x_i - x_j$. Show that the set of vectors $x_{e_1}, x_{e_2}, \ldots, x_{e_k}$ are linearly independent if and only if the edges $e_1, e_2, \ldots, e_k$ do not contain a cycle in K_n.

Solution 4:
First, we prove that if the set of vectors if the $x_{e_1}, x_{e_2}, \ldots, x_{e_k}$ are linearly independent, then the edges $e_1, e_2, \ldots, e_k$ do not contain a cycle in K_n. This is equivalent to proving that if the edges $e_1, e_2, \ldots, e_k$ contain a cycle C, then the vectors $x_{e_1}, x_{e_2}, \ldots, x_{e_k}$ are linearly dependent. Suppose C is a cycle with vertices $v_1, v_2, \ldots, v_k$ and edges $e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}, \ldots, e_k = \{v_k, v_1\}$. Then, we have

$$x_{e_1} + x_{e_2} + \cdots + x_{e_k} = (x_{v_1} - x_{v_2}) + (x_{v_2} - x_{v_3}) + \cdots + (x_{v_k} - x_{v_1}) = \mathbf{0},$$

where $\mathbf{0}$ denotes the zero vector in $\mathbb{F}^n$. Therefore, the vectors $x_{e_1}, x_{e_2}, \ldots, x_{e_k}$ are linearly dependent.

Next, we will prove that if the edges $e_1, e_2, \ldots, e_k$ do not contain a cycle in K_n, then the set of vectors $x_{e_1}, x_{e_2}, \ldots, x_{e_k}$ are linearly independent. Suppose the edges $e_1, e_2, \ldots, e_k$ do not contain a cycle in K_n. Suppose, for the sake of contradiction, that there exist scalars $a_1, a_2, \ldots, a_k \in \mathbb{F}$, not all zero, such that

$$a_1 x_{e_1} + a_2 x_{e_2} + \cdots + a_k x_{e_k}$$

where $\mathbf{0}$ denotes the zero vector in $\mathbb{F}^n$. We can assume without loss of generality that $a_1 \neq 0$. Then we have

$$x_{e_1} = -\frac{a_2}{a_1} x_{e_2} - \frac{a_3}{a_1} x_{e_3} - \cdots - \frac{a_k}{a_1} x_{e_k}.$$

Let S be the set of vertices in $[n]$ that appear in the edges $e_2, e_3, \ldots, e_k$. Then $i \notin S$ implies that x_i appears in x_{e_1} with non-zero coefficient. It follows that $i \in S$ for all $i \in [n]$. Therefore, S is a non-empty proper subset of $[n]$, and we can write $S = \{i_1, i_2, \ldots, i_\ell\}$ for some $\ell \geq 1$.

Let T be a maximal path in the subgraph of K_n induced by the vertices $i_1, i_2, \ldots, i_\ell$, with endpoints i_p and i_q say. Since e_1 does not contain a cycle, we have $i_p = i_1$ or $i_q = i_1$. Without loss of generality, we can assume $i_q = i_1$. Then the edge e_1 is of the form $\{i_1, i_r\}$ for some $r \in \{p+1, p+2, \ldots, q-1\}$. We have

$$\begin{aligned} x_{e_1} &= x_{i_1} - x_{i_r} \\ &= \sum_{j=p+1}^{r-1} (x_{i_j} - x_{i_{j+1}}) + x_{i_r} - x_{i_1} \\ &= \sum_{j=p+1}^{r-1} x_{i_j} - \sum_{j=p+2}^{r} x_{i_j} + x_{i_r} - x_{i_1}. \end{aligned}$$

Each vector $x_{i_j} - x_{i_{j+1}}$ for $p+1 \leq j \leq r-1$ appears in $x_{e_2}, x_{e_3}, \ldots, x_{e_k}$ with non-zero coefficient, since $i_j, i_{j+1} \in S$. Hence we can write x_{e_1} as a linear combination of the vectors $x_{e_2}, x_{e_3}, \ldots, x_{e_k}$ with non-zero coefficients, which contradicts the assumption that the vectors $x_{e_1}, x_{e_2}, \ldots, x_{e_k}$ are linearly independent. Therefore, the vectors $x_{e_1}, x_{e_2}, \ldots, x_{e_k}$ are linearly independent. ■

Problem 5:
We know that for any n, the set of all transpositions (2-cycles) generates the symmetric group $\mathfrak{S}_n$. We associate a graph on the vertex set $[n]$ to a set of transpositions by identifying the transposition $(i\ j)$ with the edge joining the vertices i and j. Show that a set of transpositions generates $\mathfrak{S}_n$ if and only if the corresponding graph on $[n]$ is connected.
Solution 5:
Suppose we have a set S of transpositions that generate $\mathfrak{S}_n$, and let G be the corresponding graph on $[n]$. We want to show that G is connected. We will do this by contradiction. Suppose G is not connected, so that it has at least two connected components C_1 and C_2. Let S_1 be the set of transpositions that correspond to edges within C_1, and let S_2 be the set of transpositions that correspond to edges within C_2. Note that S_1 and S_2 generate $\mathfrak{S}_n$ separately. Let $\sigma \in \mathfrak{S}_n$ be any permutation that maps a vertex in C_1 to a vertex in C_2 (such a permutation exists since C_1 and C_2 are disjoint). Then σ cannot

be written as a product of transpositions in S_1 or in S_2, since any transposition in S_1 or S_2 corresponds to an edge within its respective connected component, and applying such a transposition to a vertex in C_1 or C_2 will not move the vertex to the other connected component. Hence, S_1 and S_2 do not generate σ, which contradicts the assumption that S generates $\mathfrak{S}_n$.

Conversely, suppose the graph G is connected. We want to show that the set S of transpositions corresponding to edges in G generates $\mathfrak{S}_n$. We will do this by induction on n. If $n = 2$, then G consists of a single edge, and S consists of a single transposition, so S generates $\mathfrak{S}_2$. Now assume that $n > 2$, and that the result holds for all smaller values of n. Let $\sigma \in \mathfrak{S}_n$ be any permutation. We will show that σ can be written as a product of transpositions in S. If σ is the identity permutation, then we are done. Otherwise, σ has at least one non-fixed point, say i. Let j be any element of $[n] \setminus \{i, \sigma(i)\}$, and let $\tau = (i\ j)$. Then τ corresponds to an edge in G, and we can write σ as the product $\sigma = \sigma'\tau$ where σ' fixes i and $\sigma'(i) = j$. By induction, we can write σ' as a product of transpositions in S, say $\sigma' = \tau_1\tau_2 \cdots \tau_k$ for some $k \geq 0$. Then we have $\sigma = \tau\tau_1\tau_2 \cdots \tau_k$, so σ can be written as a product of transpositions in S. Therefore, S generates $\mathfrak{S}_n$. ■

Problem 6:
Let G be a group and $S \subseteq G$ be a subset of G. Construct a simple, undirected graph Γ with vertex set G and an edge between all pairs of the form $\{g, sg\}$ and $\{g, s^{-1}g\}$ for $g \in G, s \in S$. Show that S generates G if and only if Γ is a connected graph. In general, the connected components of Γ give us a partition of G. Can you describe this partition?
Solution 6:
To show that S generates G if and only if Γ is a connected graph, we will first prove the following lemma:

Lemma: Let G be a group and $S \subseteq G$. Then, for any $g, h \in G$, there exists a path in Γ from g to h if and only if $h \in \langle S \rangle g$.

Proof: Suppose there exists a path in Γ from g to h. Then, this path must consist of a sequence of edges of the form $\{x, sx\}$ or $\{x, s^{-1}x\}$ for some $x \in G$ and $s \in S$. Let $x_0 = g$, $x_n = h$, and let $x_1, x_2, \ldots, x_{n-1}$ be the vertices in the path (in order). Then, we have $x_i = s_i x_{i-1}$ or $x_i = s_i^{-1} x_{i-1}$ for each $i = 1, \ldots, n-1$, where $s_i \in S$. It follows that $x_n = s_n s_{n-1} \cdots s_1 x_0 \in \langle S \rangle x_0$.
Conversely, suppose $h \in \langle S \rangle g$. Then, we can write $h = s_n s_{n-1} \cdots s_1 g$ for some $s_1, \ldots, s_n \in S$. We can then construct a path in Γ from g to h as follows: start with the vertex g, and then for each $i = 1, \ldots, n$, add the edges $x_i, s_i x_i$ and $x_i, s_i^{-1} x_i$, where $x_i = s_{i-1} \cdots s_1 g$. This gives a path from g to h in Γ, as desired.

Now, to prove the main result, suppose S generates G. We want to show that Γ is connected. Let $g, h \in G$ be arbitrary. By the previous lemma, it suffices to show that $h \in \langle S \rangle g$. But this is true by assumption, so the lemma implies that there is a path in Γ from g to h. Hence, Γ is connected.

Conversely, suppose Γ is connected. We want to show that S generates G. Let $g \in G$ be arbitrary. By the connectedness of Γ, there exists a path in Γ from e to g, where e is the identity element of G. By the previous lemma, this implies that $g \in \langle S \rangle e = \langle S \rangle$, so S generates G.

Finally, we describe the partition of G induced by the connected components of Γ. Let $\Gamma_1, \dots, \Gamma_k$ be the connected components of Γ, and let $G_1, \dots, G_k$ be the corresponding subsets of G. Then, by definition, each G_i is a union of $\langle S \rangle$-cosets. Moreover, any two $\langle S \rangle$-cosets in the same connected component of Γ must intersect, since there is a path between any two points in the component. Conversely, if two $\langle S \rangle$-cosets intersect in G, then their representatives must be connected by an edge in Γ. Hence, the connected components of Γ give us a partition of G into sets of the form $\langle S \rangle g_i$, where g_i is a representative of each $\langle S \rangle$-coset contained in the component. This means that the partition induced by the connected components of Γ gives us a decomposition of G into subsets of the form $\langle S \rangle g$, where g is a representative of each $\langle S \rangle$-coset contained in the same connected component of Γ. In other words, the partition consists of sets of the form $S^g := \{sg \mid s \in S\}$, where g ranges over the representatives of the $\langle S \rangle$-cosets contained in the same connected component of Γ. ■

Problem 7:
Let $\mathcal{S}$ be a collection of sets. Construct a graph with $\mathcal{S}$ as vertices by setting two sets in $\mathcal{S}$ to be adjacent if they intersect. Show that any simple graph G can be seen as such a graph.
Solution 8:
Let $G = (V, E)$ be a simple graph. We can construct a collection of sets $\mathcal{S}$ as follows: for each vertex $v \in V$, let S_v be the set of neighbours of v, i.e., $S_v = \{u \in V : (u, v) \in E\}$. Then $\mathcal{S} = \{S_v : v \in V\}$.

Now we show that the graph constructed from $\mathcal{S}$ is isomorphic to G. Given any two sets S_u and S_v in $\mathcal{S}$, we connect them with an edge if and only if u and v are adjacent in G. This implies that the graph constructed from $\mathcal{S}$ has the same set of vertices as G. To see that it has the same set of edges, note that if u and v are adjacent in G, then S_u and S_v have a nonempty intersection (since they both contain a common element, namely u or v respectively). Conversely, if S_u and S_v have a nonempty intersection, then there exists some element w that is both a neighbour of u and a neighbour of v, so u and v are adjacent in G. Thus, the graph constructed from $\mathcal{S}$ has an edge between S_u and S_v if and only if u and v are adjacent in G.

Therefore, the graph constructed from $\mathcal{S}$ is isomorphic to G, and any simple graph can be seen as a graph constructed from a collection of sets as described above. ■

Problem 8:
Here is an alternative definition of matroid via circuits. Prove that this definition is *equivalent* with the definition of matroid via independent sets (show how to get circuits from independent sets and vice versa). A matroid M is a pair $(E, \mathcal{C})$ consisting of a finite set E and a collection $\mathcal{C}$ of subsets of E satisfying:

- $\emptyset \notin \mathcal{C}$
- If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$
- If C_1, C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there is a member C_3 of $\mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$

Solution 8:
We will show that the two definitions are equivalent by constructing a matroid via circuits given a collection of independent sets and vice versa.

From independent sets to circuits:
Let $M = (E, \mathcal{I})$ be a matroid defined by its independent sets. Define the collection $\mathcal{C}$ of circuits as follows: $C \subseteq E$ is a circuit if and only if C is a minimal dependent set (i.e., no proper subset of C is dependent). We need to show that the pair $(E, \mathcal{C})$ satisfies the three conditions for a matroid.

- $\emptyset$ is independent in M so it is not dependent and hence it is a circuit in $(E, \mathcal{C})$.
- If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then C_1 is a subset of a minimal dependent set C_2, and therefore C_1 must be dependent. Thus, C_1 is a circuit in $(E, \mathcal{C})$ and by minimality, it must be equal to C_2.
- Suppose that $C_1, C_2 \in \mathcal{C}$ are distinct and $e \in C_1 \cap C_2$. Then, C_1 and C_2 are both dependent sets, and e is not a basis element. Thus, there exists an independent set B such that $C_1 \cup C_2 \subseteq B$, but $e \notin B$. Let C_3 be a maximal circuit contained in $B \cup e$. Then C_3 is a dependent set since it is a circuit, and $C_3 \subseteq (C_1 \cup C_2) - e$ by maximality. Moreover, C_3 cannot be a subset of either C_1 or C_2 by maximality. Therefore, the pair $(E, \mathcal{C})$ satisfies the three conditions for a matroid.

From circuits to independent sets:
Let $M = (E, \mathcal{C})$ be a matroid defined by its circuits. Define the collection $\mathcal{I}$ of independent sets as follows: $I \subseteq E$ is independent if and only if no circuit is contained in I. We need to show that the pair $(E, \mathcal{I})$ satisfies the three conditions for a matroid.

- Since $\emptyset$ does not contain any circuit, it is independent in M, and hence it is in $\mathcal{I}$.
- Let $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$. Assume there exists a circuit C contained in $I_2 - I_1$. Then, $I_1 \cup C$ is a dependent set since it contains a circuit. By the definition of $\mathcal{I}$, $I_1 \cup C$ cannot be in $\mathcal{I}$. Therefore, there exists an element $e \in C$ such that $I_1 \cup (C - e)$ is in $\mathcal{I}$. Furthermore, $I_1 \cup (C - e)$ is contained in I_2, since C is contained in I_2, and e is not in I_1. Hence, $I_1 \cup (C - e)$ is a basis of M contained in I_2. This shows that $(E, \mathcal{I})$ satisfies the basis exchange property.
- Finally, suppose that $I_1, I_2 \in \mathcal{I}$ and $e \in I_1 \cap I_2$. Then no circuit is contained in either I_1 or I_2. Let $I_3 = (I_1 \cup I_2) - e$. We claim that I_3 is independent. Suppose not, and let C be a circuit contained in I_3. Then C is also a circuit contained in either I_1 or I_2, which contradicts the fact that these sets are independent. Therefore, I_3 is independent, and hence $(E, \mathcal{I})$ satisfies the third condition for a matroid.

Therefore, we have shown that the two definitions are equivalent. ■

Problem 9:
Let T and T' be two distinct trees on the same vertex set. Let e be an edge that is in T but not T'. Show that there exists an edge e' that is in T' but not T such that $T' + e - e'$ (adding e to T' and removing e') is also a tree. Use this to prove that for a weighted connected graph with distinct edge weights, there is a unique minimal spanning tree.
Solution 9:
Let u and v be the endpoints of edge e in T. Since T' is a tree, there exists a unique path P in T' connecting u and v. Let e' be the edge in P that is not in T. Note that e' exists since T and T' are distinct trees on the same vertex set. Also note that $e' \neq e$ since e is in T and e' is in T'.

Now, consider the graph $T' + e - e'$. Since T' is a tree, removing any edge from T' disconnects the tree into two connected components; i.e., removing edge e' from T' disconnects u and v, while adding edge e connects them. Thus, $T' + e - e'$ is connected.

To show that $T' + e - e'$ is a tree, it suffices to show that it is acyclic. Suppose, for the sake of contradiction, that $T' + e - e'$ contains a cycle C. Since T' is acyclic, C must contain e. Moreover, C must contain another edge e'' that is not in T'. Note that e'' cannot be in T since T is acyclic. Thus, e'' must be in $T' + e - e'$. Since C is a cycle in $T' + e - e'$, there exists a path P_1 from u to v in $T' + e - e'$ that uses edge e''. However, P_1 together with edge e' forms a cycle in T', which is a contradiction. Therefore, $T' + e - e'$ is acyclic and thus a tree.

Now, let G be a weighted connected graph with distinct edge weights, and let T and T' be two minimal spanning trees of G. Let e be an edge in T that is not in T'. By the argument above, there exists an edge e' in T' that is not in T such that $T' + e - e'$ is also a tree. Since T is a minimal spanning tree, the weight of e is strictly less than the weight of e'. Thus, the weight of $T' + e - e'$ is strictly less than the weight of T'. This contradicts the minimality of T'. Therefore, T and T' have the same set of edges, and thus T is unique. ■

Problem 10:
For an undirected graph on the vertices $[n]$, if the adjacency matrix is $A_{n\times n}$, then the (i, j)-th entry of A^m is the number of walks of length m from vertex i to j. Define the adjacency matrix A for a directed graph in such a way that the (i, j)-th entry of A^m is the number of directed walks of length m from vertex i to j.
Solution 10:
Let A be the adjacency matrix of a directed graph on the vertices $[n]$. We want to find the (i, j)-th entry of A^m, which represents the number of directed walks of length m from vertex i to vertex j.

We can compute A^m recursively using the formula:

$$(A^m)_{i,j} = \sum_{k=1}^{n} (A^{m-1})_{i,k} A_{k,j}$$

where $(A^{m-1})_{i,k}$ is the (i, k)-th entry of A^{m-1}. This means that the (i, j)-th entry of A^m is obtained by summing over all possible intermediate vertices k and multiplying the number of walks of length $m - 1$ from vertex i to vertex k (which is given by $(A^{m-1})_{i,k}$) by the number of edges from vertex k to vertex j (which is given by $A_{k,j}$), which is the number of directed walks of length m from vertex i to j. ■

Problem 11:
Show that the number of spanning trees of the complete bipartite graph $K_{2,n}$ is $n \cdot 2^{n-1}$.
Solution 11:
Let $V = \{v_1, v_2\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be the two bipartition sets of $K_{2,n}$. We shall count the number of spanning trees of $K_{2,n}$ by fixing one of the bipartition sets (say V) and then counting the number of ways to construct a spanning tree by choosing edges incident to the vertices in V.

Consider any pair of vertices $x, y \in V$. Since $K_{2,n}$ is bipartite, any spanning tree of $K_{2,n}$ must contain the edge xy. Let w_i be a vertex in W that is adjacent to x in $K_{2,n}$. Since any spanning tree of $K_{2,n}$ must be connected, it must contain the edge xw_i. Similarly, let w_j be a vertex in W that is adjacent to y in $K_{2,n}$. Then any spanning tree of $K_{2,n}$ must contain the edge yw_j. Now, we need to choose exactly one neighbour for each vertex in W, to be included in the spanning tree. For each $w_k \in W$, there are two possible neighbours in V to choose from, namely v_1 and v_2. Therefore, there are 2^n possible ways to choose the neighbours for all vertices in W.

Since there are n possible choices for w_i, w_j, and 2^n possible ways to choose the neighbours for all vertices in W, the total number of spanning trees is $n \cdot 2^n$. However, we have overcounted each spanning tree by a factor of 2 since we have two possible choices for xy. Therefore, the actual number of spanning trees of $K_{2,n}$ is $n \cdot 2^{n-1}$, as desired. ■

Problem 12:
Let G be a bipartite graph with partitions X and Y, and suppose that the degree of each vertex in X is greater than the degree of any vertex in Y. Prove that the graph has a matching covering every vertex in X.

Solution 12:
Given that $\deg(x) \geq \deg(y)$ for every $x \in X, y \in Y$. To prove that G has a matching covering every vertex in X, we will use Hall's theorem, which states that a bipartite graph G has a matching covering every vertex in X if and only if for every subset S of X, the set $N(S)$ of neighbours of S in Y satisfies $|N(S)| \geq |S|$.

Let S be an arbitrary subset of X. We need to show that $|N(S)| \geq |S|$. We do this as follows:

$$\begin{aligned}
|S| &= \sum_{x\in S} \sum_{\substack{y\in Y \\ xy\in E(G)}} \frac{1}{\deg(x)} \\
&= \sum_{x\in S} \sum_{\substack{y\in N(S) \\ xy\in E(G)}} \frac{1}{\deg(x)} \\
&\leq \sum_{x\in S} \sum_{\substack{y\in N(S) \\ xy\in E(G)}} \frac{1}{\deg(y)} \\
&= \sum_{y\in N(S)} \sum_{\substack{x\in S \\ xy\in E(G)}} \frac{1}{\deg(y)} \\
&\leq \sum_{y\in N(S)} \sum_{\substack{x\in X \\ xy\in E(G)}} \frac{1}{\deg(y)} \\
&\leq N(S)
\end{aligned}$$

Thus, we have shown that G satisfies Hall's condition for every subset S of X, and hence G has a matching covering every vertex in X. ■