Discrete Mathematics Assignment 3

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Throughout, for a statement P, we denote

$$[[P]] = \begin{cases} 1, & \text{if } P \text{ is true} \\ 0, & \text{otherwise} \end{cases}$$

Problem 1:
Suppose $P_1 = (S_1, \prec_1)$ and $P_2 = (S_2, \prec_2)$ are finite posets. Define their product poset as $P(S_1 \times S_2, \prec)$ where $(a_1, a_2) \prec (b_1, b_2)$ iff $a_1 \prec_1 b_1$ and $a_2 \prec_2 b_2$. Show that the Möbius function μ of P is $\mu((a, b), (c, d)) = \mu_1(a, c) \cdot \mu_2(b, d)$, where μ_i is the Möbius functions of $P_i, i = 1, 2$.

Solution 1:
To show that the Möbius function μ of P is $\mu((a, b), (c, d)) = \mu_1(a, c) \cdot \mu_2(b, d)$, we need to verify that the equation holds for all pairs of elements $(a, b), (c, d) \in S_1 \times S_2$ such that $(a, b) \prec (c, d)$ and is equal to 0 if $(a, b) \not\prec (c, d)$.

First, we consider the case when $(a, b) \prec (c, d)$; we then have $a \prec_1 c$ and $b \prec_2 d$, and therefore,

$$\begin{aligned}\sum_{(a,b)\prec(x,y)\prec(c,d)} \mu((x, y), (c, d)) &= \sum_{a\prec_1 u\prec_1 c} \sum_{b\prec_2 v\prec_2 d} \mu((u, v), (c, d)) \\ &= \left(\sum_{a\prec_1 u\prec_1 c} \mu_1(u, c)\right) \cdot \left(\sum_{b\prec_2 v\prec_2 d} \mu_2(v, d)\right) \\ &= \mu_1(a, c) \cdot \mu_2(b, d),\end{aligned}$$

Now, we consider the case where $(a, b) \not\prec (c, d)$. Then, there exists some (x, y) such that $(a, b) \prec (x, y) \prec (c, d)$. Since $\mu((x, y), (c, d)) \neq 0$, we have $\mu((a, b), (c, d)) = 0$, which is consistent with the formula $\mu((a, b), (c, d)) = \mu_1(a, c) \cdot \mu_2(b, d)$ since at least one of $\mu_1(a, c)$ and $\mu_2(b, d)$ is 0.

Therefore, we have shown that the formula $\mu((a, b), (c, d)) = \mu_1(a, c) \cdot \mu_2(b, d)$ holds for all pairs of elements $(a, b), (c, d) \in S_1 \times S_2$ in the product poset $P(S_1 \times S_2, \prec)$.

Problem 2:
Show that the subset poset $(2^{[n]}, \subseteq)$ is isomorphic to the boolean strings poset $(B^n, \prec)$ where $B = \{0, 1\}$ is the boolean poset and B^n is its n-fold product. Hence, derive the Möbius function of the subset poset to be $\mu(I, J) = (-1)^{|J\setminus I|}$ for $I \subseteq J$.

Solution 2:
To show that the subset poset $(2^{[n]}, \subseteq)$ is isomorphic to the boolean strings poset $(B^n, \prec)$, we will construct a bijection $\varphi : 2^{[n]} \to B^n$ that preserves the order.

Let $S \subseteq [n]$ be a subset of $[n]$. We can represent S as a binary string of length n by setting the i-th bit to 1 if and only if $i \in S$, i.e., we define $\varphi(S) = (b_1, b_2, \ldots, b_n)$, where $b_i = 1$ if $i \in S$ and $b_i = 0$ otherwise.

Clearly, φ is injective because no two distinct subsets of $[n]$ have the same binary representation. Furthermore, φ is surjective because any binary string of length n corresponds to a unique subset of $[n]$.

Now, we need to show that φ preserves the order, i.e., $S \subseteq T$ if and only if $\varphi(S) \prec \varphi(T)$. Suppose $S \subseteq T$. Then, for any $i \in [n]$, if $i \in S$, then $i \in T$, which implies that the i-th bit of $\varphi(T)$ is 1. Therefore, $\varphi(S) \prec \varphi(T)$. On the other hand, if $\varphi(S) \prec \varphi(T)$, then there exists an index $j \in [n]$ such that $\varphi(S)_j = 0$ and $\varphi(T)_j = 1$, where $\varphi(S)_j$ denotes the j-th bit of $\varphi(S)$. This implies that $j \in T$ and $j \notin S$, which implies that $S \subseteq T$.

Therefore, we have constructed an isomorphism between the subset poset $(2^{[n]}, \subseteq)$ and the boolean strings poset $(B^n, \prec)$.

Now we prove that the Möbius function of the subset poset is $\mu(I, J) = (-1)^{|J\setminus I|}$ for

$I \subseteq J$. We use induction on $k = |J \setminus I|$. If $k = 0$, then $I = J$, so by definition, $\mu(I, J) = 1$ and so the statement is true. Now let us assume that the statement is true for all non-negative integers less than k. Then for all $i \in \mathbb{N}$ satisfying $0 \leq i < k$, the interval $[I, J]$ contains $\binom{k}{i}$ elements of B_n that are $|I| + 1$ element subsets of $[n]$. If K is such a subset, then it follows from the induction hypothesis that $\mu(I, K) = (-1)^i$. Therefore,

$$\mu(I, J) = -\sum_{K \in [I,J)} \mu(I, K) = -\sum_{i=0}^{k-1} \binom{k}{i} (-1)^i = (-1)^k$$

where the last inequality follows from the binomial expansion of $(1-1)^k$, and we are done.

Problem 3:
Let $f, g : 2^{[n]} \to \mathbb{R}$ be real-valued functions on subsets of $[n]$ such that $g(J) = \sum_{I \supseteq J} f(I)$ for every $J \subseteq [n]$. Prove using the Möbius inversion formula for the subset poset that $f(J) = \sum_{I \supseteq J} (-1)^{|I \setminus J|} \cdot g(I)$.

Solution 3:
Since $g(J) = \sum_{I \supseteq J} f(I)$, we have

$$\begin{aligned}
\sum_{I \supseteq J} \mu(I, J) g(I) &= \sum_{I \supseteq J} \mu(I, J) \sum_{K \supseteq I} f(K) \\
&= \sum_{K \supseteq J} f(K) \sum_{I \subseteq K, I \supseteq J} \mu(I, J) \\
&= \sum_{K \supseteq J} f(K) [[K = J]] \\
&= f(J),
\end{aligned}$$

where the third equality follows from the fact that $\mu(I, J) \neq 0$ if and only if J is a maximal subset of I, and the fourth equality follows from the fact that the only K for which $K \supseteq J$ and $K = I$ is $K = J$.

Similar to the previous problem, here we can prove that $\mu(I, J) = (-1)^{|I \setminus J|}$ for $I \supseteq J$. Using this, we have the required equality

$$f(J) = \sum_{I \supseteq J} (-1)^{|I \setminus J|} \cdot g(I)$$

Problem 4:
For the divisibility poset $([n], \leq)$, where $a \leq b$ iff a divides b, find the Möbius function $\mu(a, b)$ for $a, b \in [n]$. Using that show that show that if $g(m) = \sum_{n|m} f(n)$, then $f(m) = \sum_{n|m} \mu_c(m/n) \cdot g(n)$, where $\mu_c(t)$ is the classical Möbius function defined as $\mu_c(t) = (-1)^k$, if t is a product of k distinct primes, for $k \geq 0$, and is defined to be zero otherwise.

Solution 4:
The Möbius function $\mu(a, b)$ for the divisibility poset $([n], \leq)$ is given by:

$$\mu(a, b) = \begin{cases} 1 & \text{if } a = b, \\ (-1)^k & \text{if } b = ap_1 \cdots p_k \text{ for distinct primes } p_1, \dots, p_k, \\ 0 & \text{otherwise.} \end{cases}$$

We prove it as follows: Let D_n be the poset over the set of divisors of n, where $a \le b$ iff $a \mid b$. Suppose $n = p_1^{k_1} \cdots p_t^{k_t}$ is the product of t primes. Then any divisor of n can be expressed as multiset $\{p_1^{k_1}, \dots, p_t^{k_t}\}$, and hence there is an isomorphism between D_n and the poset obtained by the direct product $\boldsymbol{p_1}^{k_1} \times \cdots \times \boldsymbol{p_t}^{k_t}$. Since the posets $\boldsymbol{p_i}^{k_i}$ are of linear orders, so the corresponding Möbius function is

$$\mu(p^i, p^j) = \begin{cases} 1, \text{ if } i = j \\ -1, \text{ if } i = j - 1 \\ 0, \text{ otherwise} \end{cases}$$

Given two divisors $a = \prod_i p_i^{a_i}$ and $b = \prod_i p_i^{b_i}$ with $a_i \le b_i$ for all i, then we have,

$$\mu(a, b) = \mu(\prod_i p_i^{a_i}, \prod_i p_i^{b_i}) = \prod_i \mu(p_i^{a_i}, p_i^{b_i}) = \begin{cases} (-1)^{\sum_i (b_i - a_i)}, \text{ if } b_i \in \{a_i, a_i + 1\} \\ 0, \text{ otherwise} \end{cases}$$

and the desired equality follows.

Now, let S be the set of all divisors of m. Then we have

$$\begin{aligned}
\sum_{n|m} \mu_c(m/n) \cdot g(n) &= \sum_{n \in S} \mu_c(m/n) \cdot g(n) \\
&= \sum_{n \in S} \mu\left(\frac{m}{n}, m\right) \cdot g(n) \\
&= \sum_{a, b \in S, a|b} \mu(a, b) \cdot g\left(\frac{m}{a}\right) \\
&= \sum_{b \in S} g\left(\frac{m}{b}\right) \sum_{a \in S, a|b} \mu(a, b) \\
&= \sum_{b \in S} g\left(\frac{m}{b}\right) \sum_{k=0}^{\omega(b)} (-1)^k \binom{\omega(b)}{k} \\
&= \sum_{b \in S} g\left(\frac{m}{b}\right) [[b = 1]] \\
&= f(m),
\end{aligned}$$

where $\omega(b)$ denotes the number of distinct prime factors of b.

Problem 5:
Let $P = (S, \prec)$ be an n-element poset and $x_1, x_2, \dots, x_n$ be a total ordering of S that is a linear extension of P. Let $I(P)$ denote the incidence algebra of P and recall the homomorphism φ defined in class from $I(P)$ to $n \times n$ matrices over the reals:

$$\varphi : f \mapsto M_f$$

where the (i, j)-th entry of M_f is $f(x_i, x_j)$ for $1 \le i, j \le n$. The matrix M_f is upper triangular.

- Write $M_f = D - N$ where D is a diagonal matrix and N is a strictly upper triangular matrix. That means, N is upper triangular and $N(i, i) = 0$ for all i. Show that $N^n = 0$.

- Show that $\varphi^{-1}(D)$ and $\varphi^{-1}(N)$ are defined in $I(P)$.
- Show that $\varphi^{-1}(D)$ has an inverse in $I(P)$ if and only if D is invertible.
- Suppose D is invertible. Show that $\varphi^{-1}(D^{-1}N)$ is defined, and $D^{-1}N = M$ is strictly upper triangular.
- Show that $(I-M)^{-1}$ is $I+M+M^2+\cdots+M^{n-1}$. Hence prove that if D is invertible there is a $g \in I(P)$ such that $f * g = \delta$, where $\delta \in I(P)$ is the identity element.

Solution 5:

- Since M_f is upper triangular, we can write it as $M_f = D - N$, where D is a diagonal matrix and N is a strictly upper triangular matrix, i.e., $D(i,j) = M_f(i,j)$ if $i = j$ and $D(i,j) = 0$ otherwise, while $N(i,j) = M_f(i,j)$ if $i < j$ and $N(i,j) = 0$ otherwise. To show that $N^n = 0$, we use induction on n. For $n = 1$, $N^1 = N$ is already strictly upper triangular and satisfies $N^1(i,i) = 0$ for all i.

  Now suppose $N^k = 0$ for some $k \geq 1$. Then for any $1 \leq i, j \leq n$ with $i < j$, we have
  $$(N^{k+1})(i,j) = \sum_{p=1}^{n} N^k(i,p)N(p,j) = \sum_{p=i+1}^{n} N^k(i,p)N(p,j).$$
  Since $N^k = 0$, this sum is zero, so $N^{k+1}(i,j) = 0$ for all $1 \leq i, j \leq n$ with $i < j$. Also, since N is strictly upper triangular, $N^{k+1}(i,i) = 0$ for all $1 \leq i \leq n$. Therefore, $N^{k+1} = 0$, and the claim follows by induction.

- To show that $\varphi^{-1}(D)$ and $\varphi^{-1}(N)$ are defined in $I(P)$, we need to show that there exist unique functions $g_D, g_N : S \times S \to \mathbb{R}$ such that $\varphi(g_D) = D$ and $\varphi(g_N) = N$. For g_D, we can define $g_D(x_i, x_j) = D(i,j)$ for all $1 \leq i, j \leq n$. This function is clearly well-defined and unique, and its image under φ is D.

  For g_N, we can define $g_N(x_i, x_j) = N(i,j)$ for all $1 \leq i < j \leq n$. This function is also well-defined and unique, and its image under φ is N.

  Therefore, $\varphi^{-1}(D)$ and $\varphi^{-1}(N)$ are defined in $I(P)$.

- Suppose D is invertible. We want to show that $\varphi^{-1}(D)$ has an inverse in $I(P)$ if and only if D is invertible. Suppose $\varphi^{-1}(D)$ has an inverse h in $I(P)$. Then we have $h * f = \delta$, where δ is the identity element of $I(P)$. Applying φ to both sides, we get $M_h M_f = I_n$, where I_n is the $n \times n$ identity matrix.

  Now consider the determinant of M_f. Since M_f is upper triangular, its determinant is the product of its diagonal entries, which are the values $f(x_1, x_1), f(x_2, x_2), \ldots, f(x_n, x_n)$. But these values are exactly the diagonal entries of D. Therefore, $\det(M_f) = \det(D) \neq 0$, so D is invertible.

  Now we want to show that D being invertible is a sufficient condition for $\varphi^{-1}(D)$ to have an inverse in $I(P)$. Let D^{-1} be the inverse of D. Define $g \in I(P)$ as follows: for any $x, y \in S$,
  $$g(x,y) = \begin{cases} D^{-1}, & x = y \\ 0, & x \neq y \end{cases}$$
  Then g is well-defined and belongs to $I(P)$ since D is invertible.

To show that g is the inverse of $\varphi^{-1}(D)$, we need to show that $\varphi(g * \varphi^{-1}(D)) = \delta$ and $\varphi(\varphi^{-1}(D) * g) = \delta$. We will only show the first equation since the second one follows by a similar argument.

Let $f = \varphi^{-1}(D)$. Then $M_f = D$, so M_{g*f} has diagonal entries equal to $D^{-1}D = I_n$ and all other entries equal to 0. Therefore, $M_{g*f} = I_n$ and $\varphi(g * f) = \delta$, as desired. This shows that g is indeed the inverse of $\varphi^{-1}(D)$, and so $\varphi^{-1}(D)$ has an inverse in $I(P)$.

- Suppose D is invertible. To show that $\varphi^{-1}(D^{-1}N)$ is defined, we need to show that there exists an element $f \in I(P)$ such that $\varphi(f) = D^{-1}N$.

  Let $f(x, y) = D(x, x)^{-1}N(x, y)$ for all $x, y \in S$. Then we have

$$M_f(i, j) = f(x_i, x_j) = D(x_i, x_i)^{-1}N(x_i, x_j) = (D^{-1}N)(i, j)$$

  for all $1 \leq i, j \leq n$. Therefore, $\varphi(f) = D^{-1}N$ and $\varphi^{-1}(D^{-1}N)$ is defined.

  To show that $D^{-1}N = M$ is strictly upper triangular, note that for any $1 \leq i \leq n$, we have

$$(D^{-1}N)(i, i) = \frac{1}{D(i, i)}N(i, i) = 0$$

  since $N(i, i) = 0$. Moreover, for $1 \leq i < j \leq n$, we have

$$M(i, j) = (D^{-1}N)(i, j) = \frac{1}{D(i, i)}N(i, j) = \frac{f(x_i, x_j)}{D(i, i)}$$

  since $f(x_i, x_j) = N(i, j)$, and $D(i, i) \neq 0$ since D is invertible. Therefore, $D^{-1}N = M$ is strictly upper triangular.

- Let M be as defined above, and let $S = I - M$. We will show that

$$S^{-1} = I + M + M^2 + \cdots + M^{n-1}$$

  Note that

$$S(I + M + M^2 + \cdots + M^{n-1}) = I - M^n.$$

  By the first part of this problem, we have $N^n = 0$, so

$$M^n = (D^{-1}N)^n = D^{-1}N^n = 0.$$

  Hence,

$$S(I + M + M^2 + \cdots + M^{n-1}) = I.$$

  Similarly, we have

$$(I + M + M^2 + \cdots + M^{n-1})S = I,$$

  and so

$$S^{-1} = I + M + M^2 + \cdots + M^{n-1}$$

  Since $M_f = D - N$ is invertible, we know that $I - M_f$ is also invertible. From the previous part, we have that

$$(I - M_f)^{-1} = I + M_f + M_f^2 + \cdots + M_f^{n-1}.$$

Let $g(x,y) = D(x,x)\sum_{i=0}^{n-1}(-1)^i M_f^i$ for all $x, y \in S$. Then we have

$$\begin{aligned}(f * g)(x,y) &= \sum_{z\in S} f(x,z)g(z,y) \\ &= \sum_{z\in S} f(x,z)D(z,z)\sum_{i=0}^{n-1}(-1)^i M_f^i \\ &= \sum_{i=0}^{n-1}(-1)^i \sum_{z\in S} f(x,z)D(z,z)M_f^i \\ &= \sum_{i=0}^{n-1}(-1)^i (M_f^i)(x,y) \\ &= (I - M_f)^{-1} \\ &= \delta(x,y).\end{aligned}$$

Therefore, g is the inverse of f in $I(P)$, and we have $f * g = \delta$.