Discrete Mathematics Assignment 2

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Problem 1:
Let w be any binary string of length k. For $n \geq k$, count the number of binary strings of length n that do not contain w as a subsequence.

Solution 1:
Let $f(n,k)$ denote the required number of binary strings of length n, that do not contain w as a subsequence. Suppose the first bit of w is b, then if n begins with b, the required number of binary strings is equal to the number of binary strings of length $n-1$ (first bit removed) that contains $w-b$ (b removed) as a subsequence; and if n begins with $\bar{b}$ (if $b=0$, then $\bar{b}=1$ and if $b=1$, then $\bar{b}=0$), the required number of binary strings is equal to the binary strings of length $n-1$ (first bit removed) that contains w as a subsequence. Therefore, we have the following recurrence relation:

$$f(n,k) = f(n-1,k-1) + f(n-1,k)$$

We claim that

$$f(n,k) = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k-1} \tag{1}$$

We prove it by induction on $n+k$. We have the following base cases. For $n+k=2$, we have $f(1,1)=1$. For $n+k=3$, we have $f(2,1)=1$. For $n+k=4$, we have $f(3,1)=1$ and $f(2,2)=3$, all of which satisfy equation (1). Suppose that equation (1) is true upto all $n+k-1$. Then we have,

$$\begin{aligned}
f(n,k) &= f(n-1,k-1) + f(n-1,k)\\
&= \left(\binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{k-2}\right) + \left(\binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{k-1}\right)\\
&= \binom{n-1}{0} + \left(\binom{n-1}{0} + \binom{n-1}{1}\right) + \cdots + \left(\binom{n-1}{k-2} + \binom{n-1}{k-1}\right)\\
&= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k-1}
\end{aligned}$$

and we are done. Therefore, the number of binary strings of length n that do not contain w as a subsequence is given by

$$f(n,k) = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k-1}$$

$\square$

Problem 2:
Show that the n^{th} Catalan number $\frac{1}{n+1}\binom{2n}{n}$ counts the binary strings of length $2n$ that do not contain any of the following strings as subsequences:

$$1^{n+1}, 1^n0, 1^{n-1}0^2, \ldots, 1^i0^{n+1-i}, \ldots, 10^n, 0^{n+1}$$

Solution 2:
Call a binary string *bad* if it contains one of the given strings as subsequence. We will find a bijection between bad strings and non-balanced parentheses, where 0's can be replaced with open parentheses "(" and 1's can be replaced with closed parentheses ")".

Consider a bad string of length $2n$. Assume, to the contrary, that the parentheses expression corresponding to this binary string is balanced. Since the string is bad, so it contains a subsequence of the form 1^i0^{n+1-i}. Now since the parentheses expression is balanced, so for the i closing parentheses (corresponding to 1), there are i open parentheses (corresponding to 1) before it. And similarly for the $(n+1-i)$ open parentheses, there are $(n+1-i)$ closed parentheses after it. This results to a total of $2i+2(n+1-i) = 2n+2$ parentheses which is greater than $2n$, a contradiction.

Conversely, we begin with a non-balanced parentheses expression of size $2n$. Thus, at any index k, the number of closed parentheses, at indices $\leq k$, is greater than or equal to the number of open parentheses, i.e., at any index k in the corresponding binary string representation, the number of 1's, at indices $\leq k$, is greater than or equal to the number of 0's. We shall prove that this results in a bad string. Suppose at some index, the number of 1's upto that index from the left is i, then there are at most $(i-1)$ number of 1's to the left of that index and so there are at least $n-(i-1) = n+1-i$ number of 0's to the right of that index. Thus, we have the string 1^i0^{n+1-i} as a subsequence of this string and hence it is a bad string.

Thus, we have proved the claimed bijection and since the number of non-balanced parentheses expressions of length $2n$ (which is also the same as the number of balanced parentheses expressions of length $2n$; just interchanging positions of 0's with 1's and vice versa) is equal to the n^{th} Catalan number $\frac{1}{n+1}\binom{2n}{n}$, so we are done. □

Problem 3:
Solve the following recurrence relation by relating it to a problem solved in class (or otherwise):

$$\begin{aligned} &a_0 = 1 \\ &a_n = na_{n-1} + (-1)^n \text{ for } n \geq 1. \end{aligned}$$

Solution 3:
Consider the exponential generating function of the given recurrence. It is given by

$$\varphi(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$$

where $a_n = n! \times$ coefficient of x^n in $\varphi(x)$. Now, we have,

$$\begin{aligned} \varphi(x) &= a_0 + \sum_{n\geq 1} a_n \frac{x^n}{n!} \\ &= 1 + \sum_{n\geq 1} (na_{n-1} + (-1)^n) \frac{x^n}{n!} \\ &= 1 + x \sum_{n\geq 1} a_{n-1} \frac{x^{n-1}}{(n-1)!} + \sum_{n\geq 1} (-1)^n \frac{x^n}{n} \\ &= 1 + x \sum_{n\geq 0} a_n \frac{x^n}{n!} + (e^{-x} - 1) \\ &= x\varphi(x) + e^{-x} \end{aligned}$$

Rearranging, we have

$$\varphi(x) = \frac{e^{-x}}{1-x}$$

The series expansion of e^{-x} is given by

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

and that of $\frac{1}{1-x}$ is given by

$$\frac{1}{1-x} = x + x^2 + x^3 + \cdots$$

The coefficient of x^k in the expansion of $\frac{1}{1-x}$ is 1 and that of x^{n-k} in the expansion of e^{-x} is $\frac{(-1)^k}{k!}$, and multiplying both the coefficients and taking sum over all $k \geq 0$, we get the coefficient of x^n in $\varphi(x)$. Therefore,

$$a_n = n! \sum_{k \geq 0} \frac{(-1)^k}{k!}$$

which is equal to D_n, the number of derangements on n elements. $\square$

Problem 4:
Give a combinatorial proof for the principle of inclusion exclusion

$$\left| \bigcap_{i=1}^{n} \overline{A_i} \right| = \sum_{I} (-1)^{|I|} |A_I|$$

by first moving all the negative terms in the summation to the LHS so that the equation assumes the form $A = B$, where both A and B are now sums of positive terms. Then define suitable sets whose sizes are these positive terms and give a bijective correspondence that will prove the equation.

Solution 4:
Consider a set elements (say A), each possessing a subset of properties $[n]$. Let A_I denote the set of elements having all the properties of I for any subset $I \subseteq [n]$.

Let $\mathcal{S} := \{(a, J) \mid a \in A, J \subseteq \text{set of properties of } a\}$. Call a pair (a, J) even or odd according as $|J|$ is even or odd.

Observe that for a fixed subset $I \subseteq [n]$, (a, I) is a legitimate pair if and only if $a \in A_I$.

For any $a \in A$, let s_a be its smallest property. Then the mapping $f : \mathcal{S} \to \mathcal{S}$ defined by

$$f(a, J) = \begin{cases} (a, J \cup s_a), & \text{if } s_a \notin J \\ (a, J \setminus s_a), & \text{if } s_a \in J \end{cases}$$

is a parity changing involution defined everywhere expect on pairs (a, ϕ) with $a \in A$ having no property. Then the number of odd pairs of $\mathcal{S}$ is equal to the number of even pairs of $\mathcal{S}$ which are not of the above form. Let $\mathcal{P}$ be the set of pairs (a, ϕ) for which a has no property. Thus, moving all the negative terms in the summation to the LHS and keeping the positive terms in the RHS, we get

$$\sum_{|I| \text{ odd}} |A_I| + |\mathcal{P}| = \sum_{|I| \text{ even}} |A_I|$$

Clearly, the number of pairs (a, ϕ) for which a has no property is equal to the set of elements of A having no property. Therefore,

$$|\mathcal{P}| = \left| \bigcap_{i=1}^{n} \overline{A_i} \right|$$

and hence,

$$\sum_{|I| \text{ odd}} |A_I| + \left|\bigcap_{i=1}^{n} \overline{A_i}\right| = \sum_{|I| \text{ even}} |A_I|$$

This gives the required formula

$$\left|\bigcap_{i=1}^{n} \overline{A_i}\right| = \sum_{I} (-1)^{|I|} |A_I|$$

□

Problem 5:
Solve the recurrence relation $a_r + 3a_{r-1} + 2a_{r-2} = f(r)$ where $f(r) = 1$ for $r = 2$ and $f(r) = 0$ otherwise. Assume the boundary conditions $a_0 = a_1 = 0$.
Solution 5:
Consider the generating function of the given recurrence. It is given by

$$\varphi(x) = \sum_{n \geq 0} a_n x^n$$

Now, $f(2) = 1 \implies a_2 + 3a_1 + 2a_0 = 1 \implies a_2 = 1$. Also, for $r \geq 2$, it is given that $f(r) = a_r + 3a_{r-1} + 2a_{r-2} = 0$.
Using the values of a_0, a_1, a_2 and a_n for $n \geq 3$ in the generating function, we have

$$\begin{aligned}
\varphi(x) &= x^2 + \sum_{n \geq 3} (-3a_{n-1} - 2a_{n-2}) x^n \\
&= x^2 - 3 \sum_{n \geq 3} a_{n-1} x^n - 2 \sum_{n \geq 3} a_{n-2} x^n \\
&= x^2 - 3x \sum_{n \geq 3} a_{n-1} x^{n-1} - 2x^2 \sum_{n \geq 3} a_{n-2} x^{n-2} \\
&= x^2 - 3x \sum_{n \geq 0} a_{n-1} x^{n-1} - 2x^2 \sum_{n \geq 0} a_{n-2} x^{n-2} \\
&= x^2 - 3x\varphi(x) - 2x^2 \varphi(x)
\end{aligned}$$

Rearranging terms, we have

$$\varphi(x) = \frac{x^2}{1 + 3x + 2x^2} = \frac{x^2}{(1+x)(1+2x)} = x^2 \left[\frac{-1}{1+x} + \frac{2}{1+2x}\right]$$

The series expansion of $\frac{1}{1+x}$ is given by

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \cdots$$

and that of $\frac{1}{1+2x}$ is given by

$$\frac{1}{1+2x} = \frac{1}{1-(-2x)} = 1 + (-2x) + (-2x)^2 + (-2x)^3 + \cdots$$

The coefficient of x^{n-2} in $\frac{-1}{1+x}$ is $-(-1)^{n-2} = -(-1)^n$ and the coefficient of x^{n-2} in $\frac{2}{1+2x}$ is $2(-2)^{n-2} = 2^{n-1}(-1)^n$. Therefore, we have

$$a_n = -(-1)^n + 2^{n-1}(-1)^n = (2^{n-1} - 1)(-1)^n \text{ for } n \geq 2$$

□

Problem 6:
Gossip is spread among r people via telephone. In a phone call between two people A and B, they exchange all the gossip they have heard so far. Let a_r denote the minimum number of phone calls so that all the gossip will be known to everyone. Show that $a_2 = 1, a_3 = 3, a_4 = 4$. Then show that $a_r \leq a_{r-1} + 2$. Finally, show that for $n \geq 4$, $a_r \leq 2r - 4$.

Solution 6:
For $r = 2$, one call is required and enough to share all the gossip, i.e., $a_2 = 1$. To show that $a_3 = 3$, assume there are three people P_1, P_2 and P_3. Let P_1 and P_2 share their gossip via one phone call, then if P_2 and P_3 share their gossip via one more phone call, then P_2 and P_3 get to know all the gossip. Now one more phone call, either between P_1 and P_2 or P_1 and P_3, is required and enough for all the gossip to be known to everyone. So, $a_3 = 3$. Now to show that $a_4 = 4$, assume there are four people P_1, P_2, P_3 and P_4. Let P_1 and P_2 share their gossip via one phone call and P_3 and P_4 share their gossip via another phone call. Then P_1 and P_3 can make a call and P_2 and P_4 can make another phone call so that the gossip is known to everyone. This gives $a_4 \leq 4$. We shall prove than in all other possibilities (here the order P_1, P_2, P_3, P_4 is not important), the number of phone calls is ≥ 4. Suppose P_1 and P_2 share their gossip and one of them (say P_2) shares it with P_3, then if P_3 shares it with P_1, then P_4 needs to know all the gossip of P_1, P_2, P_3 via at least one phone call and then for P_4 to pass on his gossip to others, he needs to make a number of calls exceeding 4, so we stop. Suppose P_3 shares the gossip with P_4 instead of P_1, then P_3 and P_4 get to know all the gossip, so to pass the gossip to P_1 and P_2, they need to make at least two more calls, exceeding 4. Thus, $a_4 = 4$.

Suppose a set of people $P_1, P_2, \ldots, P_{r-1}$ need a minimum of a_{r-1} phone calls (with P_1 making the first call) so that everyone knows all the gossip. Then, if a new person P_r is included, we can start with P_r making a phone call with P_1, and then the phone calls among the $r - 1$ people continues so that everyone $(P_1, P_2, \ldots, P_{r-1})$ knows all gossip in a_{r-1} phone calls. Now for P_r to know all the gossip, any one of $P_1, P_2, \ldots, P_{r-1}$ can make a call with P_r and share the gossip. So, $a_r \leq a_{r-1} + 2$.

We show that $a_r \leq 2r - 4$ by induction. We have, $a_4 = 4 = 2 \times 4 - 4$. Assuming that $a_r \leq 2r - 4$ is true for some $r \geq 4$, then by the previous result, we have

$$a_{r+1} \leq a_r + 2 \leq 2r - 4 + 2 = 2(r + 1) - 4$$

and hence by induction, we have established that $a_r \leq 2r - 4$ for $n \geq 4$. □