Discrete Mathematics Assignment 1

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Problem 1:
Starting from $\mathbb{N} \times \mathbb{N}$ is countable, which we showed in class, show that the set of all rational numbers is countable.

Solution 1:
We prove the following lemma first.

Lemma 1.1: If there exists a surjection from a countable set to a set A, then A is countable or finite.
Proof: Suppose B is countable and there exists a surjection $f : B \to A$. Then for each element $a \in A$, there exists an element $b_a \in B$ such that $f(b_a) = a$. The association $a \mapsto b_a$ is an injection because

$$a_1, a_2 \in A \text{ and } b_{a_1} = b_{a_2} \implies a_1 = f(b_{a_1}) = f(b_{a_2}) = a_2$$

Define $g : A \to B$ by $g(a) = b_a$. Then $g(A) \subseteq B$ and since B is countable, so $g(A)$ is countable or finite. Since g is a bijection between A and $g(A)$, so A is countable or finite.

Define $h : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$ by $g(m, n) = m/n$. Since every positive rational number can be written as a quotient of natural numbers, so g is surjective. Since $\mathbb{N} \times \mathbb{N}$ is countable (as showed in class), it follows from Lemma 1.1 that the set of positive rational numbers, $\mathbb{Q}^+$ is countable. Now, suppose that the positive rational numbers are enumerated as $\{0, r(1), r(2), r(3), \dots\}$. Then we can define $k : \mathbb{N} \to \mathbb{Q}$ as

$$k(n) = \begin{cases} 0, & \text{if } n = 1 \\ r(\frac{n}{2}), & \text{if } n \text{ is even} \\ -r\left(\frac{n-1}{2}\right), & \text{if } n \text{ is odd}, n \neq 1 \end{cases}$$

This is an injection because we have enumerated $\mathbb{Q}^+$ so that $r(i) = r(j) \implies i = j$. Also, it is a surjection because for any $n \neq 1$ and $r(n) > 0$, the pre-image of $r(n)$ and $-r(n)$ are $2n$ and $2n + 1$ respectively (pre-image of 0 is 1). Therefore, k is a bijection and hence $\mathbb{Q}$ is countable. $\square$

Problem 2:
If $A_1, A_2, \dots, A_k$ are countable sets then show that $A_1 \times A_2 \times \cdots \times A_k$ is countable.

Solution 2:
We use induction on k. But first we prove the following lemmas.

Lemma 2.1: If there exists an injection from a set A to a countable set, then A is countable or finite.
Proof: Suppose B is countable and there exists an injection $k : A \to B$, then we can define a bijection $k' : A \to k(A)$ by setting $k'(x) = k(x)$ for every $x \in A$. Since $k(A) \subseteq B$ and B is countable, so $k(A)$ is countable or finite. Since k' is a bijection between A and $k(A)$, so A is countable or finite.

Lemma 2.2: If A and B are countable sets, then $A \times B$ is countable.
Proof: Since A and B are countable sets, so there exist bijections $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$. We can enumerate the elements of A and B as $\{f(1), f(2), \dots\}$ and $\{g(1), g(2), \dots\}$ respectively. Then, $A \times B = \{(f(i), f(j)) \mid i, j \in \mathbb{N}\}$, which is an infinite set. Now we define $h : A \times B \to \mathbb{N}$ by $h(f(i), f(j)) = 2^i 3^j$. This is an injection because for $i_1, i_2, j_1, j_2 \in \mathbb{N}$,

$$\begin{aligned} 2^{i_1}3^{j_1} = 2^{i_2}3^{j_2} \implies i_1 = i_2, j_1 = j_2 &\implies (f(i_1), f(j_1)) = (f(i_2), f(j_2)) \\ &\implies h(f(i_1), f(j_1)) = h(f(i_2), f(j_2)) \end{aligned}$$

Therefore, using Lemma 2.1, we have $A \times B$ is countable.

Let $P_k := A_1 \times A_2 \times \cdots \times A_k$. We have, $P_1 = A_1$ is countable. Suppose that $P_{k-1} = A_1 \times A_2 \times \cdots \times A_{k-1}$ is countable. Then $P_k = P_{k-1} \times A_k$, where P_{k-1} and A_k are countable sets, and hence by the Lemma 2.2, P_k is countable. Therefore we have proved by induction that if $A_1, A_2, \dots, A_k$ are countable sets then $P_k = A_1 \times A_2 \times \cdots \times A_k$ is countable. □

Problem 3:
Is the set of all finite subsets of $\mathbb{N}$ countable? Justify your answer with proof.

Solution 3:
Yes, the set of all finite subsets of $\mathbb{N}$ is countable. Let S be the set of all finite subsets of $\mathbb{N}$. To prove this, let $A_0 = \{\phi\}$ and for $n = 1, 2, 3, \dots,$ let $A_n :=$ set of subsets of $\mathbb{N}$ whose largest element is n. Then,

$$S = \bigcup_{n=1}^{\infty} A_n$$

i.e., S is expressed as a countable union of finite sets, and hence countable.
Problem 4:
What is the dimension of the reals, $\mathbb{R}$, as a vector space over the field of rationals, $\mathbb{Q}$? Justify your answer with a proof.

Solution 4:
The dimension of $\mathbb{R}$ as a vector space over $\mathbb{Q}$, is infinite. To prove this, we assume, to the contrary, that the dimension of $\mathbb{R}$ as a vector space over $\mathbb{Q}$ is finite and let $v_1, v_2, \dots, v_n$ be a basis. Therefore, for any $x \in \mathbb{R}$, there exist unique $a_1, a_2, \dots, a_n \in \mathbb{Q}$ such that

$$x = a_1v_1 + a_2v_2 + \cdots + a_nv_n$$

i.e., the map $\mathbb{Q}^n \to \mathbb{R}$ that takes $(a_1, a_2, \dots, a_n) \mapsto a_1v_1 + a_2v_2 + \cdots + a_nv_n$ is a bijection, which implies that $\mathbb{Q}^n$ and $\mathbb{R}$ have the same cardinality. From the results of Problem 1 and 2, we have $\mathbb{Q}^n = \mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}$ (n times) is countable, but $\mathbb{R}$ is uncountable (proved in class), which is a contradiction. □

Problem 5:
Give an explicit injective map from $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Solution 5:

Any real number in $(0,1)$ can be written in the form $0.x_1x_2\dots$, where x_i's and y_i's are digits (including 0's). We define the map $f:(0,1)\times(0,1)\to(0,1)$ such that

$$f(0.a_1a_2\dots,0.b_1b_2\dots)=0.a_1b_1a_2b_2\dots$$

To make this function well-defined, we avoid decimal expansions that end with infinite successive 9's. To prove that it is injective, we consider any

$$(x,y)=(0.x_1x_2\dots,0.y_1y_2,\dots),(p,q)=(0.p_1p_2\dots,0.q_1q_2,\dots)\in(0,1)\times(0,1)$$

Then

$$\begin{aligned}f(x,y)=f(p,q)&\implies 0.x_1y_1x_2y_2\dots=0.p_1q_1p_2q_2\dots\\&\implies x_i=p_i,y_i=q_i\ \forall i\in\mathbb{N}\implies (x,y)=(p,q)\end{aligned}$$

Therefore, f is injective. Now, as in Problem 6, define $g:(0,1)\to\mathbb{R}$ such that

$$g(x)=\tan\left(\pi\left(x-\frac{1}{2}\right)\right)$$

which is bijective (as proved in Solution 6). Therefore, we can define the required function as $h:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that

$$h(x,y)=g\circ f\Big(g^{-1}(x),g^{-1}(y)\Big)$$

which is injective as f is injective and g,g^{-1} are bijective. □

Problem 6:
Give an explicit bijection from the open interval $(0,1)$ to the reals $\mathbb{R}$.

Solution 6:
We define $g:(0,1)\to\mathbb{R}$ such that

$$g(x)=\tan\left(\pi\left(x-\frac{1}{2}\right)\right)$$

Then for $x\neq\frac{1}{2}$,

$$g'(x)=\pi\sec^2\left(\pi\left(x-\frac{1}{2}\right)\right)>0$$

and for $x=\frac{1}{2}$, $g'(x)=0$ and hence g is strictly increasing in the interval $(0,1)$. Therefore, g is injective. Also, every $y\in\mathbb{R}$ has a pre-image x such that

$$x=\frac{1}{\pi}\tan^{-1}(y)+\frac{1}{2}$$

which should lie in $(0,1)$. This is clear because the range of $\tan^{-1}(y)$ is $(-\frac{\pi}{2},\frac{\pi}{2})$ and hence

$$0=\frac{1}{\pi}\cdot\left(-\frac{\pi}{2}\right)+\frac{1}{2}<x<\frac{1}{\pi}\cdot\frac{\pi}{2}+\frac{1}{2}<1$$

Therefore, g is also surjective and hence g is the required bijection. $\square$

Problem 7:
What is the cardinality of the set of all continuous functions from $\mathbb{R} \to \mathbb{R}$? Justify with proof.

Solution 7:

We denote by $\mathcal{C}^0$, the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. For every $c \in \mathbb{R}$, we can define a function $f_c : \mathbb{R} \to \mathbb{R}$ such that for each $x \in \mathbb{R}$, $f_c(x) = c$ so that the association $c \mapsto f_c$ is an injection. Therefore, $|\mathbb{R}| \leq |\mathcal{C}^0|$. Now let $\{q_1, q_2, \dots\}$ be an enumeration of the rational numbers and let $\mathcal{S}$ be the set of sequences of real numbers. Then the association $f \mapsto \{f(q_n)\}_{n=1}^{\infty}$ is an injection. This is because for any $x \in \mathbb{R}$, $\exists \ \{x_m\} \in \mathbb{Q}$ such that $x_m \to x$ as $m \to \infty$ (this is possible because $\mathbb{Q}$ is dense in $\mathbb{R}$). Therefore, if f, g are continuous, then $f(x_m) \to f(x)$ and $g(x_m) = g(x)$ as $m \to \infty$. Thus, $f(r_m) = g(r_m) \ \forall m \implies f(x) = g(x)$ (by uniqueness of limit) and hence we proved that the above association is an injection. Therefore, $|\mathcal{C}^0| \leq |\mathcal{S}|$. Let $\mathcal{F}_{\mathbb{N}}$ be the set of all functions from $\mathbb{N}$ to $\mathbb{R}$, then we have $|\mathcal{S}| = |\mathcal{F}_{\mathbb{N}}|$. This is because given any sequence $\{a_n\}_{n=1}^{\infty}$, we can just define $f(n) = a_n$. Since every real number can be written as a binary expansion, so we have $\mathcal{F}_{\mathbb{N}}$ has the same cardinality as that of the set of functions from $\mathbb{N}$ to (the set of functions from $\mathbb{N}$ to $\{0, 1\}$), which has the same cardinality as that of the set of functions from $\mathbb{N} \times \mathbb{N}$ to $\{0, 1\}$, which has the same cardinality as that of the set of functions from $\mathbb{N}$ to $\{0, 1\}$. But since every real number can be written as a binary expansion, so the of the set of functions from $\mathbb{N}$ to $\{0, 1\}$ is same as that of $\mathbb{R}$. Therefore, $|\mathcal{C}^0| \leq |\mathbb{R}|$. Hence, by Cantor-Schroeder-Bernstein's theorem, we get $|\mathcal{C}^0| = |\mathbb{R}|$, i.e., the cardinality of the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ is same as that of $\mathbb{R}$.

Problem 8:
What is the cardinality of the set of all bijections from $\mathbb{N} \to \mathbb{N}$? Justify your answer with a proof.

Solution 8:

Let S be the set of all bijections from $\mathbb{N}$ to $\mathbb{N}$. Consider the set

$$P := \{(2n - 1, 2n) : n \in \mathbb{N}\}$$

For each subset $A \subseteq P$, define $f_A : \mathbb{N} \to \mathbb{N}$

$$f_A(n) = \begin{cases} n - 1, & \text{if } n \text{ is odd and an element of some pair of } A \\ n + 1, & \text{if } n \text{ is even and an element of some pair of } A \\ n, & \text{if } k \notin \bigcup A \end{cases}$$

Clearly, there exists a bijection between P and $\mathbb{N}$, so $|P| = |\mathbb{N}| = \aleph_0$. Now, each of the $2^{\aleph_0}$ subsets of $\mathbb{N}$ defines a distinct bijection f_A from $\mathbb{N}$ to $\mathbb{N}$. Therefore,

$$2^{\aleph_0} \leq |S|$$

Also, any function from $\mathbb{N}$ to $\mathbb{N}$ is a subset of $\mathbb{N} \times \mathbb{N}$ and hence

$$|S| \leq 2^{\aleph_0}$$

Therefore, by Cantor-Schroeder-Bernstein's theorem, we have

$$|S| = 2^{\aleph_0} = |\mathbb{R}|$$

i.e., the cardinality of the set of all bijections from $\mathbb{N}$ to $\mathbb{N}$ is equal to the cardinality of $\mathbb{R}$.$\square$

Problem 9:
Derive the well-ordering theorem assuming Zorn's lemma.

Solution 9:

We first state Zorn's lemma and well-ordering theorem.

Zorn's lemma: Let $(P, \leq)$ be any poset such that every chain in P has an upper bound in P, then P has a maximal element.

Well-ordering theorem: Any non-empty set can be well-ordered.

Let S be any non-empty set and let

$$\mathcal{A} := \{(A, \leq) : A \subseteq S, \ \leq \text{ is a well-ordering on } A\}$$

be the collection of pairs $(A, \leq)$, where $A \subseteq S$ and $\leq$ is a well-ordering on A. Define a relation $\preceq$ on $\mathcal{A}$ so that for all $x, y \in \mathcal{A}$, $x \preceq y$ if and only if x equals an *initial segment* of y (if $(A, \leq)$ is well-ordered, then the set $\{a \in A : a \leq k\}$ is called an initial segment of A). This is reflexive, transitive and anti-symmetric, since one set is an initial segment hence a subset of the other. Therefore the relation $\preceq$ defines a partial order relation on $\mathcal{A}$. For each chain $C \subseteq \mathcal{A}$, define $C' = (R, \leq')$, where R is the union of all sets A for all $(A, \leq) \in C$ and $\leq'$ is the union of all relations $\leq$ for all $(A, \leq) \in C$. Then, C' is an upper bound for C in $\mathcal{A}$. Therefore, by Zorn's lemma, $\mathcal{A}$ has a maximal element, say $(M, \leq_M)$. We claim that M contains all the members of S. This is true because if not, then for any $a \in S \setminus M$, we can construct $(M', \leq_{M'})$, where $M' = M \cup \{a\}$ and $\leq_{M'}$ is extended so that a is greater than every element of Y. Then $\leq_{M'}$ defines a well-order on (M') and $(M', \leq_{M'})$ would be larger than $(M, \leq_M)$, a contradiction. Therefore, since M contains all the elements of S and $\leq_M$ is a well-ordering on M, it is also a well-ordering on S. $\square$

Problem 10:
Derive the axiom of choice assuming Zorn's lemma.

Solution 10:

We state the axiom of choice first.

Axiom of choice: Let S be any set of non-empty sets $S = \{S_i\}_{i \in I}$. Then there is a function $f : S \to \bigcup_{i \in I} S_i$ such that $f(S_i) \in S_i$ for all $i \in I$.

Let S be any non-empty set. Consider pairs (T, f) consisting of a subset $T \subseteq S$ and a choice function f on T. We introduce a partial order on the set of all such pairs by defining $(T, f) \preceq (T', f')$ whenever $T \subseteq T'$ and $f'|_T = f$. The poset is non-empty because for every $a \in S$, there is an obvious partial choice function on $\{a\}$. For every chain C in this poset, we can define $T^* = \bigcup_{(T,f)\in C} T$ and $f^*(S) = f(S)$ for any S such that f is defined on S. Then (T^*, f^*) is an upper bound for C. Therefore, by Zorn's lemma, there is some maximal element, say (M, f_M). If $b \in S \setminus M$, then we can extend f from M to $M \cup \{b\}$ by defining $f(A) = b$ for any A containing b, which contradicts maximality and so $S \setminus M = \phi$. Therefore, f_M is a choice function for M and hence for S. $\square$