Discrete Mathematics (Midsem Exam)

Feb 23, 2023, 10:00 AM to 12:30 PM
Total: 100 points

1. Let A and B be infinite sets and $f : A \to B$ a surjection such that $|f^{-1}(b)|$ is *finite* for all $b \in B$. Show that $|A| = |B|$. **10 points**

2. Suppose A is a countable subset of reals. Show that there is a real number a such that $(a + A) \cap A = \emptyset$, where $a + A = \{a + b \mid b \in A\}$. **10 points**

3. We saw in class that the number of balanced parenthesis strings of length $2n$ is the n^{th} Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Show using a bijective argument that the number of non-balanced parenthesis strings of length $2n$ is $\binom{2n}{n+1}$. Hence show that $C_n = \frac{1}{n+1}\binom{2n}{n}$. **10 points**

4. Find a closed form for the generating function $g(x)$ for the sequence $a_n = n^2, n \geq 1$. Obtain from it the generating function for the sequence $b_n = \sum_{i=1}^{n} i^2$ and hence a closed form for $\sum_{i=1}^{n} i^2$. **10 points**

5. A university has 35000 first-year students who have to do 4 courses each out of 999 possible course choices. The maximum size of lecture rooms is 135. Can the students choose courses to meet this constraint? Justify your answer with an argument. **10 points**

6. 17 people correspond by mail with one another - each one with all the rest. In their letters only 3 different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least 3 people who write to each other about the same topic. **10 points**

7. An *increasing* binary tree with n nodes is a binary tree with n nodes labeled from $\{1, 2, \ldots, n\}$ such that the labels are in increasing order along any path from the root. Show that the number of increasing binary trees on n nodes is $n!$ by giving a bijection to all permutations on $\{1, 2, \ldots, n\}$. **10 points**

8. Let $\text{fix}(\pi)$ denote the number of fixed points of a permutation $\pi \in S_n$ (i.e. number of cycles of length 1 in the product of disjoint cycles representation of π). Show that $\sum_{\pi \in S_n} \text{fix}(\pi) = n!$ by giving a combinatorial proof for it, by giving a bijection between S_n and a set of size $\sum_{\pi} \text{fix}(\pi)$. (You can use ideas from Lagrange's theorem in group theory to construct the bijection.) **10 points**

9. Find the closed form for the exponential generating function for Stirling numbers of the second kind, $S(n, k)$ for fixed k. **10 points**

10. Let A and B be $(n+1) \times (n+1)$ matrices with rows are columns indexed by $0, 1, \ldots, n$ such that $A_{ij} = \binom{i}{j}$ and $B_{ij} = (-1)^{i+j}\binom{i}{j}$. Show using ideas from the inclusion-exclusion principle that the matrix product $AB = I$. **10 points**

1