Discrete Mathematics (Endsem Exam)

April 25, 2023, 9:30 AM to 12:45 PM
Total: 120 points

1. In a directed graph, a king is defined as a node from which all the other nodes are reachable in at most two steps. A tournament is a directed graph that has exactly one edge between any distinct pair of vertices. Prove that every tournament has a king. Can there be more than one king in a tournament?

8+2=10 points

2. The spanning tree game is a 2-player game. Each player in turn selects an edge. Alice starts by deleting an edge, and then Bob fixes an edge (which has not been deleted yet); an edge fixed cannot be deleted later on by the other player. Bob wins if he succeeds in constructing a spanning tree of the graph; otherwise, Alice wins. Prove the following. If there are two spanning trees in the graph whose edge sets are disjoint, Bob can win no matter how Alice plays.

Hint: You may use the following fact that you proved in assignment 4. Let T and T' be two distinct trees on the same vertex set. Let e be an edge that is in T but not T'. There exists an edge e' that is in T' but not T such that $T'+e-e'$ (adding e to T' and removing e') is also a tree. **10 points**

3. An undirected graph can be converted into a directed graph by choosing a direction for every edge. This is called an *acyclic* orientation of a graph.

(a) Show that for every undirected graph, there is a way of choosing directions for its edges so that the resulting directed graph has no directed cycles.

(b) For a cycle with 3 vertices, you can see that there are 6 possible acyclic orientations. Show that for K_n (the complete graph with n vertices), there are $n!$ different acyclic orientations.

4+6=10 points

4. Use Hall's marriage theorem for bipartite matching to prove the following.

(a) (Existence of a system of distinct representatives/traversal) Let $A_1, A_2, \ldots, A_n$ be some *finite* sets. Assume that the union of any p of these sets has size $\geq p$, for all $p \in \{0, 1, \ldots, n\}$. (In other words, assume that $A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_p} \geq p$ for any $1 \leq i_1 < i_2 < \ldots < i_p \leq n$. Then, we can find n distinct elements $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

(b) An $r \times n$ matrix consisting of the numbers $1, \ldots, n$ is called a latin rectangle if the same number never appears twice in any row or any column. Let $r < n$. Any $r \times n$ latin rectangle can be extended to an $(r+1) \times n$ latin rectangle by suitably adding a row.

5+5=10 points

5. Given any graph, you don't have to remove more than half the edges from it to make it bipartite. Prove this by showing the following claim. Any graph G has a bipartite subgraph H with $V(H) = V(G)$ and $\text{DEG}_H(v) \geq \text{DEG}_G(v)/2$ for all $v \in V(G)$. **10 points**

6. The incidence matrix of an undirected graph has rows indexed by the vertices and columns indexed by the edges. The (i, j)-th entry is 1 if the vertex v_i is incident on the edge e_j. It is 0 otherwise. Let A be the incident matrix of an undirected graph G. Let $E' = \{e_{i_1}, \ldots e_{i_k}\}$ be a subset of edges of the graph G. Show the following using induction on k.

(a) If the edges in E' do not contain a cycle, then the columns in A associated with those edges are linearly independent.

(b) Show that the converse is true if the graph is bipartite. The converse may fail for non-bipartite graphs, for example, K_3. Can you suggest a modification for the incidence matrix to make the converse work for all graphs?

5+5=10 points

7. You have seen the deletion and contraction method to prove the matrix tree theorem and to get a recurrence for chromatic polynomials. Here is another application: Let the number of acyclic orientations (defined in problem 3) of a multigraph be $\kappa(G)$.

If G contains a loop $\kappa(G)$ is 0 and if G has no edges it is defined to be 1. For all the edges e, show that $\kappa(G) = 2\kappa(G/e)$ if e is a cut-edge (here, G/e denotes contraction) and $\kappa(G) = \kappa(G/e) + \kappa(G-e)$ otherwise.

10 points

8. Suppose there are n letters and their corresponding n addressed envelopes. Suppose the letters are randomly placed inside the envelopes one by one: pick a letter and randomly pick an empty envelope for it. What is the probability that exactly k letters are correctly placed in their envelopes? Justify your answer with complete proof. **10 points**

9. A *transposition* is a 2-cycle $(i\ j)$ for $i \neq j$. I.e. it is a permutation on $\{1, 2, \ldots, n\}$ that swaps i and j and fixes all other elements. Show that a product of transpositions $(a_i\ a_j)$, where $a_i, a_j \in \{1, 2, \ldots, n\}$, is an n-cycle if and only if the pairs $\{a_i, a_j\}$ are the edges of a tree on n vertices. **10 points**

10. Explain Möbius inversion in a poset, defining all the terms involved. Let X be a finite set and $(\text{EVEN}, \subseteq)$ be the poset of all even cardinality subsets of X ordered by inclusion. Derive the Möbius function for this poset. **10 points**

11. Let $c : \mathcal{R}^2 \to [k]$ be a k-coloring of the points on the plane. Show, using the pigeon-hole principle that there must be a rectangle whose vertices are monochromatic. Can you generalize this to 3-dimensions? If so, then state and prove it. **10 points**

12. A family $\mathcal{F}$ of subsets of $[n] := \{1, \ldots, n\}$ is said to be intersecting if $A \cap B \neq \emptyset$ for all sets $A \in \mathcal{F}$ and $B \in \mathcal{F}$. Show that it is possible to have an intersecting family containing 2^{n-1} subsets of $[n]$. Also, show the maximum cardinality of an intersecting family is $\leq 2^{n-1}$. **10 points**